"Quantum optics"

WS 2016-2017 Worksheet 12

While the mathematical framework of quantum mechanics was essentially fully developed in the 1920's, some of the implications continue to give pause to physicists. One early hope of scientists was that quantum mechanics would prove to be an intermediate theory, where the to be uncovered deeper theory would be able to remove the apparently strange aspects of quantum mechanics. An important result was due to J. S. Bell. He was able to show that any possible future theory, which reproduces the predictions for measurement outcomes given by quantum mechanics, cannot be reconciled with all assumptions of classical physics. Therefore, quantum mechanics makes predictions, which differ from classical physics that can be experimentally tested. In the current exercise sheet, we will study an illustrative example of Bell's theorem (1964) as well as derive a refined version of a Bell inequality due to Clauser, Horne, Shimony and Holt (1969). The paper work is about an experimental violation of the CHSH-Bell inequality and quantum teleportation.

Exercise 28: Bell inequality

Imagine you encounter the following experimental setup: there are two identical detectors with three measurement settings (let us label them A, B, C), as well as a red and a green lamp. The two detectors are facing each other and are separated by a considerable distance. Half way between the two detectors is a box, which emits a signal, when a button is pushed. The signal will be detected at both detectors simultaneously, causing either the red or the green light to flash. You start to experiment with different detector settings. You observe that whenever both detectors share the same setting, e.g. both are set to A, both detectors flash the same light, either green or red. However, it seems impossible to tell which detector will flash red or green in advance. If the two detectors do not share the same settings, you will see all combinations between red and green flashes of the two detectors rather randomly. A possible explanation of your observation is that both detectors receive perfect copies of the same signal while the three settings of the detector correspond to three different properties of the signal to be measured. You further assume, that for each detector setting, which light will flash is already determined when the signal is sent from the box[1].

1) Let us assume that the machine decides which signal to send based on a joint probability distribution of the three physical properties A, B, C with $\sum_{A,B,C\in\{r,g\}} P(A,B,C) = 1$. As you will be only able to determine 2 out of 3 properties of the signal, you will only be able to determine joint probabilites of two of these observables[2]. Use the fact that the joint probability of two observables can be obtained from the full joint probability distribution to show that the following inequality holds.

$$P(A,B|A=B) + P(A,C|A=C) + P(B,C|B=C) \geq 1 \tag{1}$$

2) Based on your reasoning, you run a set of experiments to find out $P(A,B|A=B) = P(A,C|A=C) = P(B,C|B=C) = 1/4$, therefore Eq. (1) adds up to $P(A,B|A=B) + P(A,C|A=C) + P(B,C|B=C) = 3/4 < 1$. Want can you conclude about that fact?

[1] We thus assume that we can assign values to observables before we measure them. This assumption is often called realism or counterfactual definiteness

[2] Here we assume that the measurement settings on one detector will not alter the measurement results on the other detector. This assumption is called locality or separability in the literature

3) Before you continue, think about what possible arguments could be made to circumvent your conclusion from the previous exercise? Can you come up with modifications of the set up to refute these arguments?

4) A possible physical realization of a setup as discussed here can be implemented in the following way. The signal emitted from the box corresponds to two two-level systems or qubits in the Bell state $|\phi^+\rangle = 1/\sqrt{2}\,(|0\rangle|0\rangle + |1\rangle|1\rangle)$. The three settings on the detector correspond to spin projection along three different directions. Let the three observables be characterized by the following set of eigenvectors

$$\begin{aligned} &A : |a_0\rangle = |0\rangle, |a_1\rangle = |1\rangle; \quad B : |b_0\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, |b_1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle; \\ &C : |c_0\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, |c_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2} - \frac{1}{2}|1\rangle \end{aligned} \tag{2}$$

Write down the Bell state $|\phi^+\rangle$ in the eigenbasis of each of the three observables. What can you conclude about measurements with equal detector settings?

5) Finally, calculate the probabilities $P(A, B|A = B), P(A, C|A = C), P(B, C|B = C)$. (Hint: write the Bell state $|\phi^+\rangle$ in the following way: for the qubit on which you want to measure observable A, express its single particle state in the basis of eigenvectors of A, etc..)

Exercise 29: CHSH Inequality

The treatment of the Bell inequality in the previous exercise relied on the fact that the measurement outcomes are perfectly correlated if the settings of the detector are equal. A possible verification of the inequality is thus quite susceptible to imperfections in the measurement setup. Clauser, Horne, Shimony and Holt developed a generalisation of the original Bell inequalities, which circumvents this restriction. In the following we assume that two parties, let us call them Alice and Bob, have each excess to one part of a bipartite system. Alice and Bob can perform measurements on their respective subsystem only. For simplicity, we assume that they each have the choice to measure one of two observables, which can each assume the values $\{\pm 1\}$. We will refer to Alice two observables as a, a' and Bobs observables as $\{b, b'\}$ respectively.

1. Convince yourself that under these circumstances the following equality holds

$$C = (a + a')b + (a - a')b' = \pm 2 \tag{3}$$

2. Alice and Bob now choose to measure either of their accessible observables and write down their measurement result. Imagine that Alice and Bob have access to multiple copies of the system, so that they can repeat the experiment several times and they afterwards share their results. Starting from Eq. 3, show that the following inequality holds for the measurements of Alice and Bob

$$|\langle ab\rangle + \langle a'b\rangle + \langle ab'\rangle - \langle a'b'\rangle| \leq 2 \tag{4}$$

3. In quantum mechanics, the correlation functions given above are calculated according to

$$\langle \hat{a}\hat{b}\rangle = \langle \phi|\hat{a}\hat{b}|\phi\rangle, \tag{5}$$

given that the system is in the pure state $|\phi\rangle$. Calculate the correlation functions appearing in equation 4. Let the state shared between Alice and Bob be the Bell state $|\psi^-\rangle = 1/\sqrt{2}\,(|0\rangle|1\rangle - |1\rangle|0\rangle)$. The observables are given as $a = \vec{a}\cdot\vec{\sigma}$, where $\vec{a}$ is a 3-dimensional unit vector and $\vec{\sigma}$ is a 3 dimensional vector whose ith component is the ith Pauli-Matrix. For which choices directions $\vec{a}, \vec{a'}, \vec{b}, \vec{b'}$ is the inequality 4 maximally violated?

Exercise 30: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres
Hensen *et al.*, Nature **526**, 682-686 (2015)

Experimental quantum teleportation
D. Bouwmeester *et al.*, Nature **390**, 575-579 (1997).