1. Introduction

1.1 The Schrödinger equation

(brief reminder) \[ \text{Heat radiation} \xrightarrow{\text{Planck spectrum}} \frac{\omega^3}{e^{h\omega/kT} - 1} \]

Planck, Einstein: energy quantized for light

\[ E = \hbar \omega \]

\[ \hbar = \frac{h}{2\pi} \approx 10^{-34} \text{Js} \]

\( \hbar: \) "Planck's quantum"

\[ \text{de Broglie} \]

Einstein: photon momentum

\[ p = \hbar k \]

\[ k = \frac{2\pi}{\lambda} \]

(note: goes together with \( E = \hbar \omega \) due to relativity)

\[ \text{de Broglie: particles \rightarrow matter waves?} \]

also with \( E = \hbar \omega, \ p = \hbar k \)

Quit now

\[ E = \frac{p^2}{2m} \]

instead of \( E = pc \)

[actually, de Broglie used the relativistic energy

\[ E = \sqrt{m^2c^4 + p^2c^2} \] ]
Goal: derive linear wave equation that yields correct dispersion relation \( c_0 = c_0(k) \)

\[
E = \frac{p^2}{2m} \\
\hbar c_0 = \frac{\hbar^2 k^2}{2m}
\]

In free space: no \( x \) or \( t \) special \( \Rightarrow \) solutions will be plane waves

\[
\psi_{\text{plane}}(x, t) = e^{i(kx - \omega t)}
\]

\[
i \hbar \partial_t \psi = \frac{\hbar}{2m} \partial_x^2 \psi \]

\[
- \hbar \partial_x \psi = \frac{\hbar k}{i} \psi
\]

\(
\Rightarrow \text{use:}
\)

\[
i \hbar \partial_t \psi = \frac{-i \hbar \partial_x^2}{2m} \psi + V(x) \psi
\]

Schrödinger equation 1926

alternative & equivalent:
Heisenberg's "matrix mechanics" 1925

compatible with
\( E = \frac{p^2}{2m} + V(x) \)

(& keeps conservation of \( \hbar^2 k^2 \), see below)

"Standing waves" yield discrete energy levels

Ansatz \( \psi_n(x, t) = \phi_n(x) e^{-i \frac{\hbar}{\epsilon} E_n t} \)

with \( \hat{H} \phi_n = E_n \phi_n \) \( \uparrow \) energy eigenvalues
and with the "Hamiltonian operator"
\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(x)
\]
\[
\hat{p} = -i\hbar \frac{\partial}{\partial x}
\]
momentum operator

Remember:
\( E_n \) are real and \( \phi_n \) can be chosen as orthonormal basis

\[
\langle \phi_n | \phi_m \rangle = \int \phi_n^*(x) \phi_m(x) dx = \delta_{n,m}
\]

Scalar product between vectors of Hilbert space

Examples:

- Particle in a box
  \[
  E_n = \frac{n^2 \hbar^2}{2mL^2}
  \]
  \( n = 1, 2, 3, ... \)

- Harmonic oscillator
  \[
  E_n = \hbar \omega (n + \frac{1}{2})
  \]
  \( n = 0, 1, 2, ... \)
  "number of quanta"
  (plenty of applications, especially in quantum field theory!)

- Hydrogen atom
  \[
  E_n = -\frac{E_1}{n^2}
  \]
  \( n = 1, 2, 3, ... \)
  \( E_1 = \frac{m_e^4}{8\hbar^2 \varepsilon_0^2} = 13.6 \text{eV} \)

Many particles:
\[
\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(x_1, x_2)
\]
with \( \Psi(x_1, x_2) \)

and so on, for more particles
Impressive Success of QM:
explained...
- atoms & molecules & chemical bonds & crystals
- colors of materials (absorption, heat radiation)
- electrical conductivity, magnetism,
  mechanical properties
- nuclear structure, radioactivity, fission & fusion,
  elementary particles
- light quanta

understood/predicted new effects/applications...
- superconductors & superfluids
- lasers
- nuclear & electron spin magnetic resonance
- semiconductors (→ transistors, computers)

future: maybe room-temperature superconductors?
maybe quantum computers?

> 2/3 of physics research today
needs QM directly!
1.2 The meaning of \( \psi \)?

Compare other wave fields:

- Sound waves: pressure, density \( p \)
- Elastic waves, surface waves: displacement field \( \delta \)
- Electromagnetic waves: electric field & magnetic field \( E, B \)

("what is moving?" \( \rightarrow \) "aether?"
\( \rightarrow \) no! relativity!)

(a) Conserved density: probabilities!

Linear wave eq. \( \Rightarrow \) expect conserved quantities
quadratic in wave field

"Local conservation" \( \Rightarrow \) find density \( S \) & current density \( \mathcal{J} \),
such that

\[
\frac{\partial S}{\partial t} + \text{div} \mathcal{J} = 0
\]

"Equation of Continuity"

[hydrodynamics: \( S = S_{\Omega} \)]

Claim: For the Schrödinger eq.,

\[
S_{\Omega} = |\psi(x)|^2
\]

is a conserved density, with

\[
\mathcal{J}(x) = \text{Re} \left[ \frac{\psi^*(x) \frac{-i}{m} \nabla \psi(x)}{m} \right]
\]

Proof: \( \frac{\partial S}{\partial t} = \ldots \) use SEQ \( \Rightarrow \checkmark \)

Note: This \( S, \mathcal{J} \) are independent of \( \hat{A} \)

--- In contrast to local energy density

\[
S_{\varepsilon} = \text{Re} \left[ \frac{\psi^*(x) \hat{A} \psi(x)}{m} \right]
\]
First guess: \( e |\psi|^2 = \) charge density of electron, smeared out
(Schrödinger)

* Sounds OK for atoms & molecules: microscopic!
* Is used today for molecular structure & motion

... but:
* Should different parts of cloud repel each other?
* Wave can become extended!

Example (Born): Scattering of \( e^- \) from atom (Franck-Hertz experiment)
\( e^- \) wave extended over metres!

New interpretation (Born, 1926):
\[ |\psi|^2 = \text{probability density} \]
\[ |\psi(x)|^2 \ dx_1 dx_2 dx_3 = \text{probability to find } e^- \text{ in volume} \]
\[ \Rightarrow 2 \nabla \cdot \text{div } \psi = 0 \text{ becomes conservation of probability!} \]

In general, unitary time-evolution
\[ |\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle \]
with
\[ \hat{U}(t) = \hat{U}(t)^\dagger \Rightarrow \langle \psi(t) | \psi(t) \rangle = \langle \psi(0) | \hat{U}(t)^\dagger \hat{U}(t) | \psi(0) \rangle = \langle \psi(0) | \psi(0) \rangle \]
Statistical ideas before 1926:

- Statistical mechanics:
  - Maxwell-Boltzmann distribution
  - Brownian motion...

- Radioactive decay → elusive individual stochastic behavior!

- Einstein's ideas about photons (intensity of light ≈ average photon density)

Actual single-quantum measurements at the time:

- In radioactive decay (fluorescence, Geiger-Müller counter)
- Cloud & bubble chambers for high-energy particles

→ showed individual events,
  but no quantum interference

Around 1926, only ensemble measurements were known for quantum phenomena

Today:

- Photon detectors (e.g. avalanche photodiode, s.c. photon detectors,...)
- Electron detectors
- Also detect single atoms/ions
The "collapse of the wavefunction"

Q: How does \( \Psi \) evolve after detection at \( \bar{x} = \bar{x}_0 \)?
A: Replace \( \Psi \) by new wavefunction, localized around \( \bar{x}_0 \).

"collapse"

\[
\text{spreading of } \Psi \quad ("\text{ballistic expansion}"
\]

General rule (von Neumann's projection postulate):
measure observable \( \hat{A} \) (\( \hat{A} \) is Hermitian operator)

\[ \Rightarrow \text{obtain eigenvalue } A_n \text{ with probability } \left| \langle \Phi_n | \Psi \rangle \right|^2 \]

\[ \Rightarrow \text{"collapse" into new state } \Psi \rightarrow \Phi_n \]

"Describes most measurements"

Disadvantages:
- "Ad hoc" postulate, outside of S EQ
- Artificial distinction between quantum system & "classical measurement apparatus"
- Can we describe m snt within S EQ?
- What about msnts with any partial information? ("weak msnts")

\[ \Rightarrow \text{See modern theory of msnts!} \]
(c) "What happens really at the level of single particles?"

Example: emission of $e^-$ (or photon) from atom

Possible interpretation: emission into a random direction

random classical trajectories. OK for this case ✓
($\approx$ Einstein's thinking about photons around 1905)

But: This naive idea does not work for interference setups!
Example: Double-slit setup

If each \( e^- \) has definite trajectory:

- each \( e^- \) goes through only one slit

\[ \Rightarrow \text{(naive) consequences:} \]

- pattern on screen should lie \( 50\% \) from upper trajectories (e.g.) \( 50\% \) from lower

- pattern from upper traj. alone can lie aligned by closing lower slit (and vice versa)

\[ \Rightarrow \text{contradict expriment:} \quad \text{"which-way expr."} \]

- no interference if any slit closed

- interference pattern if both slits are open

Naive idea yields \( |\Psi_u|^2 + |\Psi_l|^2 \)

But we actually observe

\[ |\Psi_u + \Psi_l|^2 \]

\[ = |\Psi_u|^2 + \Psi_u^* \Psi_l + \Psi_l^* \Psi_u + |\Psi_l|^2 \]

depends on relative phase
Heisenberg's explanation of "Heisenberg microscope":

- Measurement of particle position to accuracy $\Delta x$ randomizes momentum, with
  $$\Delta p \geq \frac{\hbar}{2\Delta x}$$

- If $\Delta x < \text{slit distance} \implies$ this is enough to destroy interference pattern!

Lesson:
- Observation of quantum particles may strongly perturb their behavior!
  (Not unexpected for microscopic particles!)

- Perturbation is so strong that we can never observe trajectory without destroying interference effects!

$\implies$ Copenhagen interpretation (Bohr et al.):
- No trajectories in QM!
- Particle position (or momentum, etc.) becomes real only upon measurement!
Many-particle wave functions

\[ \Psi(x) \rightarrow \Psi(x_1, x_2, x_3, \ldots, x_N) \]

\[ \int |\Psi(x_1, \ldots, x_N)|^2 \, dx_1 \, dx_2 \ldots dx_N = \text{probability to find this configuration} \]

(compare classical statistical physics! \( S(x_1, \ldots, x_N) \))

Challenge for interpretation of \( \Psi \):
  - Waves in configuration space?
  - Measurement of \( x_1 \) seems to affect \( x_2 \)
### 1.3 Experimental progress in the past 80 years

<table>
<thead>
<tr>
<th>1925</th>
<th>today</th>
</tr>
</thead>
<tbody>
<tr>
<td>atomic structure: only indirect evidence — frequencies and intensities of transitions</td>
<td></td>
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<tr>
<td>weak excitation of many atoms (e.g. spectroscopy in a gas)</td>
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<tr>
<td>interference experiments only on ensemble of particles (absence intensity)</td>
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<tr>
<td>observe natural quantum systems</td>
<td></td>
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<tr>
<td>See individual e⁻ orbitals (AFM, STM) &amp; pictures of atoms &amp; molecules on surfaces</td>
<td></td>
</tr>
<tr>
<td>excite atoms strongly, 100% in excited state, detect state for single atom, observe quantum jumps</td>
<td></td>
</tr>
<tr>
<td>detect individual quanta (e⁻, photons etc.), produce single quanta &amp; do interference experiments</td>
<td></td>
</tr>
<tr>
<td>design, fabricate, control coherently artificial quantum systems</td>
<td></td>
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<tr>
<td>&gt;2/3 of physics research needs quantum mechanics!</td>
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2. Bell's inequalities and entanglement

Is there a theory that "explains" QM?

2.1 Strange correlations

(following Mami, 1985)

Idea: Bring out "strange" aspects of nature we refer to "Gedanken" Experiment:

"detector" with a knob of 3 settings

"Source" emits something (upon pressing button)

After emission, each detector flashes one of its lights.

Observations:

1. Whenever settings are the same, the same color is found at both detectors. (Even if settings are chosen just at the "last second" before lights flash!)

2. If settings are chosen completely randomly & independently, \[ P(11) = P(12) = \ldots = \frac{1}{3} \] then the colors are uncorrelated \[ P(RR) = P(GG) = P(RG) = P(GR) = \frac{1}{4} \].

Should you be bothered?
(1) How to get these correlations?

- Not just: "my flash R" \( \rightarrow \) would contradict (2.)

- Radio signal or similar between detectors?
  No! If settings are chosen \( \text{such that just at before flash} \quad c \cdot \Delta t < \text{distance (A-B)} \)
  Combination of settings not available to detectors!

- Possibly: due to source emitting correlated objects (particles, waves, ...)
  \( \Rightarrow \) joint properties!
  Example: Emit identical objects
  \( \begin{array}{ccc}
  \text{Detect} & \text{Shape, Color, Size} \\
  \text{Detect moment, etc.} \\
  \end{array} \)
  In each run: each object should know
  in advance which color to flash
  for any setting

  Example: \( \begin{array}{ccc}
  1 & 2 & 3 \\
  R & R & G \\
  \end{array} \) "instruction set",
  the same for both objects

Why?
- Cannot know which setting will be chosen, but must show same color
  if it's the same setting (\& no signalling!)
  \( \Rightarrow \) need results for all settings

- No randomness allowed at detection, due to (1.)
  (only in choosing instruction set)

Note: This "instruction set" is a property
  of detector-object
  (but separately for A \& B)
(2.) If \( \text{RRR} \) or \( \text{GGG} \) : never RG or GR as results

If anything like \( \text{RGG}, \text{GRG} \) etc.:

for random settings \( P(\text{RG}) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} < \frac{1}{4} = \frac{2}{8} \)

\[ \Rightarrow \text{We never get } P(\text{RG}) = \frac{1}{4} \]

(regardless of instruction sets or random choice thereof)

\[ \Rightarrow (1.) \text{ has enforced too much of a tendency towards some color} \]

Conclusion:

Only reasonable way to guarantee (1.) is incompatible with (2.)!

Ways out:

- Maybe there is no except like this? (There is!)
- Supraliminal signaling?
  - Allows joint instruction sets:
    
    | RR | 12 | 21 |
    |----|----|----|
    | GR |    |    |
"Is QM the result of some underlying theory?"

Example: Classical mechanics $\rightarrow$ Statistical Physics $\rightarrow$ Thermodynamics, Hydrodynamics

Problem: $x, p$ cannot be known simultaneously, "no trajectories" $\Rightarrow$ ?

Einstein, Podolsky, Rosen (1935): Want to construct situation, where both $x$ and $p$ of a particle can be determined simultaneously with certainty!

(& since QM does not describe these "elements of reality", it is an "incomplete" theory)

How to circumvent Heisenberg's uncertainty principle?

$\Rightarrow$ Trick: Use two particles, in state

$$\Psi(x_1, x_2) \sim S(x_1-x_2)$$

$$= \int \frac{dp}{2\pi\hbar} \int \frac{dp}{2\pi\hbar} e^{i\frac{p_1(x_1-x_2)}{\hbar}}$$

wave function
for $p_1=p$, $p_2=-p$

$\Rightarrow$

(1) $x_1=x_2$ in any measurement

(2) $p_1=-p_2$, """" $p_1, p_2$

Now do the following:

measure $x_1 \Rightarrow$ deduce value that $x_2$

would have in a measurement (namely: $x_2=x_1$)

but now get value of $p_2$ by measurement!
Why possible? Because most of $x_1$ cannot suddenly have changed state of distant particle! (unlike Heisenberg's reasoning for single particle!)

$\Rightarrow$ Both $x_2$ & $p_2$ "real"

Summary:

- No spooky action at a distance
- QM describes results of sept in 2 particles

EPR

QM "incomplete", more detailed description possible = "hidden variable theory"

Bohr's reply: You are not allowed to treat particles separately, every choice of most combinations (like "$x_1/p_2$", "$p_1/p_2$" etc.) corresponds to a different experiment, deduction of "what would have been" is not allowed.
Schrödinger (1935):

EPR works because

$$\Psi(x_1, x_2) \neq \text{product } \Phi_1(x_1) \cdot \Phi_2(x_2)$$

$$\Rightarrow "\Psi \text{ is entangled}"$$

["Urschrankung"]

Bohm's version of EPR (1951):

Two spin $\frac{1}{2}$ particles

each: $|\uparrow\rangle$ and $|\downarrow\rangle$ states
(eigenstates of $\hat{S}_z$, with $\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$ etc.)

$$\begin{array}{c}
\hat{S}_z \\
|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle |\downarrow\rangle_2 - |\downarrow\rangle \uparrow_2) \\
\Rightarrow \text{entangled!}
\end{array}$$

$$\Rightarrow \text{e.g.: } \hat{S}_{z_1} |\Psi\rangle = \frac{\hbar}{2} \cdot \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2)$$

measure Spin 1 along $\hat{\sigma}_z$ $\Rightarrow$

- $|\uparrow\rangle_1 \Rightarrow |\Psi\rangle = |\uparrow\rangle_1 |\downarrow\rangle_2$ after measurement
- probability $50\%$ $\Rightarrow |\downarrow\rangle_1 \Rightarrow |\Psi\rangle = |\downarrow\rangle_1 |\uparrow\rangle_2$ 

$\Rightarrow$ subsequent measurement of Spin 2 along $\hat{\sigma}_z$:

- exactly opposite!

(analogous to "$p_1 = -p_2"\)
Reminder: Spin

Orbital angular momentum:

\[ \hat{L} = \hat{r} \times \hat{p} \]

\[ \hat{L}^2 \] has eigenvalues \( \hbar^2 (l+1) \)

\[ l = 0, 1, 2, \ldots \]

\[ [\hat{L}_x, \hat{L}_y] = 0 \]

\( \hat{L}_z \) has eigenvalues \( \pm m \)

\( m = -l, \ldots, +l \)

→ express \( \hat{L}_{x,y,z} \) in 2\( l+1 \)-dim. subspaces

\( \{1, 3, 5, \ldots\} \)

Example: \( p \)-orbitals in hydrogen atom

\[ m = -l, 0, l \]

Simplest case: Spin \( \frac{1}{2} \) \( (l = "s\) = \( \frac{1}{2} \)\)

\[ \hat{S}_x, \hat{S}_y, \hat{S}_z \]

\[ \hat{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Pauli spin matrices (in eigenbasis of \( \hat{S}_z \))

Eigenstates of \( \hat{S}_z \):

\[ \hat{S}_z |\uparrow\rangle = + |\uparrow\rangle \]

\[ \hat{S}_z |\downarrow\rangle = - |\downarrow\rangle \]
Eigenstates of $\hat{\mathbf{S}}_x$:

\[ |\rightarrow\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \]
\[ |\leftarrow\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle) \]

\[ \hat{\mathbf{S}}_x |\rightarrow\rangle = + |\rightarrow\rangle \]
\[ \hat{\mathbf{S}}_y |\leftarrow\rangle = - |\leftarrow\rangle \]

Likewise: $\hat{\mathbf{S}}_y$ eigenstates

\[ +1 : \quad \frac{1}{\sqrt{2}} (|\uparrow\rangle + i|\downarrow\rangle) \]
\[ -1 : \quad \frac{1}{\sqrt{2}} (i|\uparrow\rangle + |\downarrow\rangle) \]

Spin expectation:

\[ \langle \hat{\mathbf{S}} \rangle = \text{unit vector for single spin } \frac{1}{2} \]

"Bloch vector"

(Note: completely determines state, if we are dealing with single spin $\frac{1}{2}$.)
Projection of spin on arbitrary direction:

\[ \hat{n} \cdot \hat{S} = \frac{1}{2} \hat{n} \cdot \hat{\mathbf{z}} \]

unit vector

\[ (\hat{n} \cdot \hat{\mathbf{z}})^2 = n_x^2 \hat{\mathbf{z}}_x^2 + \ldots + n_x n_y \frac{(\hat{\mathbf{z}}_x \hat{\mathbf{z}}_y + \hat{\mathbf{z}}_y \hat{\mathbf{z}}_x)}{\hat{n}} \equiv 0 \]

\[ = \hat{n}^2 = 1 \]

\[ \Rightarrow \hat{n} \cdot \hat{\mathbf{z}} \text{ has eigenvalues } \pm 1 \text{ (just like any spin projection)} \]

Rotations:

from angular momentum:

\[ \hat{L}_z = -i \hbar \frac{\partial}{\partial \phi} \]

\[ \Rightarrow [e^{-i \phi \frac{\partial}{\partial \phi}}, \psi(r, \theta, \phi)] = \psi(r, \theta, \phi - \phi) \]

rotated by \( \phi \) around \( z \)-axis

\[ e^{-i \phi \frac{\partial}{\partial \phi}} = e^{-i \frac{\phi}{\hbar} \hat{L}_z} = \text{rotation around } z \]

more general:

\[ e^{-i \frac{\phi}{\hbar} \hat{\mathbf{L}}} \]

\[ [\hat{L}_x, \hat{L}_y] \neq 0 \text{ implies that rotations around different axes do not commute} \]

Generalization to spin:

\[ e^{-i \frac{\phi}{\hbar} \hat{S}} \]
Note:
\[ \hat{R}(\phi) = e^{-\frac{i}{2} \frac{\phi}{\hbar} \hat{\mathbf{S}}_z} = e^{-\frac{i}{2} \frac{\phi}{\hbar} \hat{\mathbf{S}}_z} = \exp \left( \frac{\phi}{2} \right) \mathbf{1} - i \sin \left( \frac{\phi}{2} \right) \cdot \hat{\mathbf{S}}_z \]

⇒ full rotation, \( \phi = 2\pi \):
\[ \hat{R}(\phi = 2\pi) = -\mathbf{1} \]

⇒ changes sign of spin!
(true for any half-integer spin)
⇒ has been measured in experiment!

Two Multiple Spin \( \frac{1}{2} \):

Product Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \)

Product basis, e.g.: \( |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \equiv |\uparrow\downarrow\rangle \)

\[ \hat{S}_{2z} |\uparrow\downarrow\rangle = (\hat{S}_{2z} \otimes 1_2) |\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle \]

\[ = +\frac{h}{2} |\uparrow\downarrow\rangle \]

\[ \hat{S} = \hat{S}_1 + \hat{S}_2 \quad \text{total spin} \]

\[ \hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2 \hat{S}_1 \hat{S}_2 \]

Eigenstate for \( \hat{S}^2 \) with eigenvalue 0:

"Singlet" state
\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]

\[ \hat{S} |\psi\rangle = 0 \]

⇒ \[ e^{-\frac{i}{\hbar} \frac{\phi}{2} \hat{\mathbf{S}}} |\psi\rangle = |\psi\rangle \]

rotationally invariant!
Measure $\hat{S}_x$ for Spin 1 & 2 ⇒

$$P_{\rightarrow\rightarrow} = |\langle \rightarrow_{1}\rightarrow_{2} | \psi \rangle|^2$$

We have: $|\rightarrow_{1}\rightarrow_{2} \rangle = \frac{1}{2} (|\uparrow_{1}\uparrow_{2} \rangle + |\uparrow_{1}\downarrow_{2} \rangle + |\downarrow_{1}\uparrow_{2} \rangle + |\downarrow_{1}\downarrow_{2} \rangle)$

$$\langle \rightarrow_{1}\rightarrow_{2} | \psi \rangle = \frac{1}{2} \psi \cdot \frac{1}{\sqrt{2}} (1-1) = 0$$

$$P_{\rightarrow\leftarrow} = |\langle \rightarrow_{1}\leftarrow_{2} | \psi \rangle|^2$$

We have: $|\rightarrow_{1}\leftarrow_{2} \rangle = \frac{1}{2} (|\uparrow_{1}\uparrow_{2} \rangle - |\uparrow_{1}\downarrow_{2} \rangle + |\downarrow_{1}\uparrow_{2} \rangle - |\downarrow_{1}\downarrow_{2} \rangle)$

$$\langle \rightarrow_{1}\leftarrow_{2} | \psi \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot (-1-1) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow P_{\rightarrow\leftarrow} = \frac{1}{2}$$

Likewise $P_{\leftarrow\rightarrow} = \frac{1}{2}$ and $P_{\leftarrow\leftarrow} = 0$

$$\Rightarrow S_x \text{ measurement results are also exactly opposite! } (\text{just like } S_z)$$

Alternative approach: project onto eigenstates subsequently

$$|\psi \rangle \rightarrow |\rightarrow_{1}\leftarrow_{1} \rangle \psi \rangle \rightarrow |\rightarrow_{1}\leftarrow_{1} \rangle \leftarrow_{1} \rightarrow_{1} \rangle \psi \rangle$$

$$= |\rightarrow_{1} \rangle \cdot \frac{1}{\sqrt{2}} |\leftarrow_{2} \rangle$$
2.3 Bell's inequalities

Q: Can we find underlying theory to explain EPR-expt. (in Bohm's version)?

First, naive attempt:

Each spin = unit vector
\[ \vec{A} \uparrow^\alpha \quad \cdots \quad \vec{B} \uparrow^\beta \]
\[ \vec{\alpha} \text{ random (uniform over sphere)} \]

Msrmt. on \( A \), along direction \( \vec{\alpha} \)

\[ \Rightarrow \text{Result } "\uparrow" \text{ if } \vec{\alpha} \cdot \vec{\alpha} > 0 \]

\[ A(\vec{\alpha}, \vec{\alpha}) = \text{sign}(\vec{\alpha} \cdot \vec{\alpha}) \]

Msrmt. result: \( \pm 1 \) for \( \uparrow/\downarrow \)

Likewise:

\[ B(\vec{b}, \vec{\alpha}) = -\text{sign}(\vec{\alpha} \cdot \vec{b}) \]

\[ \Rightarrow \text{ If } \vec{\alpha} = \vec{b}, \text{ then always } \]

\[ A(\vec{\alpha}, \vec{\alpha}) = -B(\vec{\alpha}, \vec{\alpha}) \]

\[ \Rightarrow \text{ Opposite spin directions } \checkmark \]

Just like QM!

If \( \vec{\alpha} \perp \vec{b} \):

\[ \vec{A} \cdot \vec{B} > 0 \text{ as often as } \vec{A} \cdot \vec{B} < 0 \]

\[ \Rightarrow \langle AB \rangle = \int d\vec{\alpha} \text{ S}(\vec{\alpha})A(\vec{\alpha}, \vec{\alpha})B(\vec{\alpha}, \vec{\alpha}) = 0 \]

No correlations, like QM! \( \checkmark \)
Bell's important new idea:

Check also $\vec{a} \neq \vec{b}$ and $\vec{a} \times \vec{b}$

other angles between $\vec{a}$ and $\vec{b}$!

In this model:

$\vec{a} \cdot \vec{b} = 0$

$\theta = \pm (\vec{a}, \vec{b})$

$\text{sign}(\vec{a} \cdot \vec{a}) = \text{sign}(\vec{a} \cdot \vec{b})$ in a fraction

$\frac{2\theta}{2\pi} = \frac{\theta}{\pi}$ of cases

$\Rightarrow \quad \langle AB \rangle = (+1) \cdot \frac{\theta}{\pi} + (-1) \cdot (1 - \frac{\theta}{\pi})$

$\quad \quad = (1 - 2\frac{\theta}{\pi})$ (for negative $\theta$; $\theta \mapsto |\theta|$)
\[
\langle AB \rangle_{\text{QM, singlet state}} = \langle \Psi | (\hat{a} \cdot \hat{\mathbf{b}}_1) (\hat{b} \cdot \hat{\mathbf{b}}_2) | \Psi \rangle = -\hat{a} \cdot \hat{b} = -\cos \Theta
\]

\[
\begin{array}{c}
\uparrow \\
\rightarrow
\end{array}
\]

If we reproduce perfect correlations, we fail for other angles!

Is this a general rule?
Define a general \textit{local hidden variable (LHV) model} for the EPR(Bohm) - Experiment:

Hidden variable(s) \( \lambda \)
Probability density \( S(\lambda) \)
Measurement results \( A(\bar{a}, \lambda) \in \{+1, -1\} \)
and \( B(\bar{b}, \lambda) \in \{+1, -1\} \)
for detector settings \( \bar{a}, \bar{b} \) (= meas axis)
Local because we do not allow \( A(\bar{a}', \bar{b}', \lambda) \)

\( \Rightarrow \) Constraints for statistics ?
Consider \( \langle AB \rangle = \int d\lambda \ S(\lambda) A(\bar{a}, \lambda) B(\bar{b}, \lambda) = E(\bar{a}, \bar{b}) \)
Look at several combinations of settings \( (\Rightarrow \text{both for all } \bar{b} \text{ and other angles!}) \)
& get joint constraint!

Bell's original version: 2 directions at each spin \( (\bar{a}, \bar{b}; \bar{a}', \bar{b}') \)
Let \( A(\bar{a}', \lambda) = A' \) etc.

Let \( +\bar{a}' = \bar{b}' \) and assume perfect
(anti-) correlations are observed \( \Rightarrow \)
\( A' = -B' \)
(for "almost all" \( \lambda \),
for each \( \lambda \) i.e. except for a
set of measure zero)
\[ |A'B' - AB| = |A(B' - B)| = |B' - B| = |A' + B| = 1 + A'B \]

\[ A = \pm 1 \quad B = -A' \quad A' = \pm 1 \quad B = \pm 1 \]

Now \[ |X| \geq |\langle X \rangle| \Rightarrow \]

\[ |\langle AB' \rangle - \langle AB \rangle| \leq 1 + \langle A'B \rangle \]

Bell's inequality 1964

Obeyed by every LHV (that shows perfect anticorrelations for \( -a' = b' \))

Compare with QM, \( \langle AB \rangle = -\hat{a} \cdot \hat{b} \) etc.:

Choose:
(for example)

\[ \Rightarrow \langle AB' \rangle_{QM} = -\frac{1}{\sqrt{2}} \quad , \quad \langle AB \rangle = 0, \quad \langle A'B \rangle = -\frac{1}{\sqrt{2}} \]

\[ \Rightarrow |\langle AB' \rangle - \langle AB \rangle| = \frac{1}{\sqrt{2}} \]

\[ 1 + \langle A'B \rangle = 1 - \frac{1}{\sqrt{2}} \]

But \[ \frac{1}{\sqrt{2}} \neq 1 - \frac{1}{\sqrt{2}} \]

because \[ 1 - \sqrt{2} - 1 \approx 0.4 \]
Generalization (Clauser, Holt, Horne, Shimony 1969)

\[ |A B + A B'| + |A'B - A'B'| \]

\[ \leq |B + B'| + |B - B'| \]

\[ \leq 2 \]

if only values \( \pm 1 \) & 0

"no detection" results (finite detector efficiency)

& \[ |x| \geq |<x>| \]

\[ \Rightarrow \]

\[ |<AB> + <AB'>| + |<A'B> - <A'B'>| \leq 2 \]

CHHS inequality

for any LHV

with \( A \in \{0, +1, -1\} \)

and no further assumptions (don't need \( B' = -A' \) etc.)

Note: Locality is there because the value of \( A = A(\bar{a}, 2) \) is assumed to be independent of the other setting, i.e.

\[ A \bar{B} = A(\bar{a}, 2) B(b, 2) \]

\[ A \bar{B}' = A(\bar{a}, 2) B(b', 2) \]

the same \( A \)!

CHHS is the version used in modern analysis!
CHHS:

Compare with QM (for spin singlet)

Choose

\[
\begin{array}{c}
\text{a'} \\
\text{b'} \\
\text{a} \\
\text{b}
\end{array}
\]

\[
\Rightarrow \quad <AB> = -\cos(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}
\]

\[
<AB'> = -\cos(\frac{3\pi}{4}) = \pm \frac{1}{\sqrt{2}}
\]

\[
<AB''> = -\cos(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}
\]

\[
<AB'''> = -\cos(\frac{3\pi}{4}) = +\frac{1}{\sqrt{2}}
\]

\[
\Rightarrow \text{ LHS of CHHS:}
\frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.8 > 2
\]

\[
\Rightarrow \text{ conflict between QM \& LHV !}
\]

\[
\Rightarrow \text{ Test in experiments!}
\]
Note: Mermin's "gedanken" experiment

'Same settings → same colors'
'Random " → random colors''

Works for spin $\frac{1}{2}$ if settings ≠ measurement axis in the following way:

(A) 3
     /\ 120°
 1 - 2 -

(B) 2
     /\ 1
 3 - 1

(→ exercises!)
2.4 Bell test experiments (for EPR/Bohm)

Overview: Possible systems

He-atom

\[ E \]

ground state:
Spin singlet of two \( e^- \)
(fermions \( \Rightarrow \) symmetric orbital wave funct.)
in principle: double ionization \( \Rightarrow \) singlet pair

similar \( e^- \) singlet states:
other atoms, quantum dots \( \uparrow \uparrow \)
or even Fermi sea (in metal)

\( S=0 \) for even number of \( e^- \)

Cooper pairs in a superconductor

(Note: need careful extraction of singlet pairs out of many-body state!)

Nuclear spins in molecules

\[ H_2 \]
proton spins in singlet = "para-hydrogen"
triplet = "ortho-hydrogen"
"para" preferred at low \( T \)
difference in rotational states (protons need total antisymmetric \( \Psi \))

modern version on \( Hg_2 \)
in progress (mercury dimer)
[\(^{199}Hg \) has only nuclear spin \( \frac{3}{2} \)]
Singlets from scattering

Two fermions \((\text{spin } \frac{1}{2})\) \(\Rightarrow\) total \(\uparrow\downarrow\) antisymmetric

\[ \Rightarrow \]

<table>
<thead>
<tr>
<th>Spin part</th>
<th>Orbital part</th>
</tr>
</thead>
<tbody>
<tr>
<td>singlet (S=0) [ \frac{1}{\sqrt{2}} (</td>
<td>\uparrow\uparrow\rangle -</td>
</tr>
<tr>
<td>triplet (S=1) [ \frac{1}{\sqrt{2}} (</td>
<td>\uparrow\uparrow\rangle +</td>
</tr>
</tbody>
</table>

\[ \Rightarrow \]

for "short-range"* interaction: mostly singlet state is scattered!* compared to \(\uparrow\downarrow\)

In general: uncorrelated spins before scattering \(\Rightarrow\) correlated spins after " (depending on scattering angle)

\[ \Rightarrow \] "low energy"* proton-proton scattering expts

\[ * \sim \text{MeV} \]
Photons from positronium annihilation

Photon "spin" = polarization of electromagnetic wave
e.g. horizontal |H>, vertical |V>

Result for this case:

\[ \frac{1}{\sqrt{2}} (|HV\rangle - |VH\rangle) \]

alternatively: circular polarization
right-handed |R>, left-handed |L>

\[ \frac{1}{\sqrt{2}} (|RL\rangle - |LR\rangle) \]

zero total spin

→ Wu & Shabnov 1950 experiment
Note: Polaimation of light

(first, classical):
Consider
\[ E(t) = \text{Re} \left[ E_0 e^{i(\mathbf{H} \cdot \mathbf{r} - \omega t)} \right] \]
\[ E_0 = \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow E(t) = \begin{pmatrix} \cos(\omega t) \\ -i \sin(\omega t) \end{pmatrix} \]
\[ \Rightarrow \]

Let's call this "right-handed"
(Warning: opposite definitions are also used!)

\[ \Rightarrow \ QM: \text{let } |R\rangle = \frac{|H\rangle + i|V\rangle}{\sqrt{2}} \]
\[ |L\rangle = \frac{|H\rangle - i|V\rangle}{\sqrt{2}} \]

\[ \Rightarrow \ \frac{1}{\sqrt{2}} (|R\rangle - |L\rangle) = \ldots = \frac{i}{\sqrt{2}} (|VH\rangle - |HV\rangle) \]

Note:

Spin $\frac{1}{2}$

\[ \langle AB \rangle = -\cos \frac{\pi}{4}(\hat{a}, \hat{b}) \]
\[ \text{for } \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle) \]

Light

\[ \langle AB \rangle = -\cos 2\pi(\hat{a}, \hat{b}) \]
\[ \text{for } \frac{1}{\sqrt{2}} (|HV\rangle - |VH\rangle) \]
Photons from atomic cascade

Simple picture:
in an $S \leftrightarrow p_x$ transition,
the dipole moment oscillates
along $x$:

$$\langle p_x | \frac{\mathbf{p}}{4} | S \rangle \parallel x$$

for single $e^{-}$
(otherwise $\sum_j \hat{\mathbf{p}}$
)

here: coherent superposition of both pathways

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |HH\rangle + |VV\rangle \right)$$

$|\psi\rangle$ is independent of choice for
linear polarization axis:

basis transformation: $|H\rangle = c \ |H'\rangle + s \ |V\rangle$
$|V\rangle = c \ |V'\rangle - s \ |H'\rangle$

$$\Rightarrow |\psi\rangle = \cdots = \frac{1}{\sqrt{2}} \left( |H'H'\rangle + |V'V'\rangle \right)$$

Note: "Local" unitary transformation applied
to photon 2 can turn this into singlet
state form $|H_2\rangle \mapsto |V_2\rangle$, $|V_2\rangle \mapsto -|H_2\rangle$
Photons from parametric down-conversion

\[ P \sim E + E^2 + E^3 + \ldots \]  
(polarization)

energy conservation: \( \hbar \omega = \hbar \omega_{1} + \hbar \omega_{2} \)
momentum conservation: \( \hbar \vec{k} = \hbar \vec{k}_{1} + \hbar \vec{k}_{2} \)

Quantum description:
\[ \hat{H} = \sum_{k} \hbar \omega_{k} \hat{a}_{k}^{\dagger} \hat{a}_{k} + \sum_{k,k_{1},k_{2}} \hbar g_{k,k_{1},k_{2}} \hat{a}_{k_{1}}^{\dagger} \hat{a}_{k_{2}} \hat{a}_{k} + \text{h.c.} \quad (+\ldots) \]

\( k = (\vec{k}, \vec{\omega}) \)  
wave vector  
polarization index (two values)

\[ \hbar = \hbar_{1} + \hbar_{2} \]  
matrix element for transition includes constraint

entanglement in:
- energy
- momentum
- polarization  
\( \text{depends on details of material} \)

often: select energy & directions \( \Rightarrow \text{keep only polarization entanglement} \)

\[ \frac{1}{\sqrt{2}} (\text{HV} + 1\text{VH}) \]
Qubits

Control: manipulate & prepare entangled state via pulses & interaction

Examples:

- Trapped ions

Superconducting qubits

Example:

\[ \left\{ \begin{array}{c} \uparrow \uparrow \Rightarrow \\ \downarrow \downarrow \Rightarrow \text{interaction energy} \end{array} \right. \]

\[ \text{switch on drive at } \omega \text{ for well-defined time} \Rightarrow \]

\[ \text{produce } \frac{1}{\sqrt{2}} (|\uparrow \downarrow \rangle + |\downarrow \uparrow \rangle) \]

(no term with \( |\uparrow \uparrow \rangle \))

Other example:

Interaction \( |\uparrow \downarrow \rangle \leftrightarrow |\downarrow \uparrow \rangle \]

\[ \Rightarrow \text{Excite 1st qubit, wait} \]

\[ |\Psi \rangle = \frac{|\uparrow \downarrow \rangle - i|\downarrow \uparrow \rangle}{\sqrt{2}} \]
History

- positronium decay: 1950 & 70s
- proton scattering (Lambe - Rachtig & Mifflig '76)
- First cascade experiments in 70s
  1972 Freedman & Clauser (first one)
  Calcium

\[
\begin{align*}
4p^3 \, ^1S_0 & \rightarrow 4p4s \, ^3P_1 \\
3d4p \, ^3P_1 & \rightarrow 4p4s \, ^3P_1 \quad \text{lifetime } \sim 5 \text{nsec} \\
& \Rightarrow \gamma_1 \text{ & } \gamma_2 \\
& \text{emitted within this time!} \\
& \Rightarrow \text{coincidence counting!}
\end{align*}
\]

Frequency filters to distinguish \( \gamma_1 \) / \( \gamma_2 \)

Polarization filters:

Only one polarization detected!

Overall Detector efficiency \( \sim 10^{-3} \) (including solid angle)

Dark count rate \( \sim 100 \text{ Hz} \)
Collect only "back-to-back" photon pairs

\[ \Rightarrow \text{Coincidence rate } \geq 0.1 \text{ Hz} \]
\[ \text{accidental } \leq 0.01 \text{ Hz} \]

\[ \sim 200 \text{ h data collection!} \]

After accounting for inefficiencies:
- agree with QM
- violate 'suitably modified' Bell inequality
  \[ \text{deviation } \approx \text{ "4 standard deviations" } \]
Aspect et al., 1980s

- Ca cascade
- First to use both polarization channels
- First to switch fast the polarization direction 1982

Bragg reflection on ultrasound wave in water → periodic switching at ~50 MHz

⇒ "space-like" separation of detection

here: L/c ~ 40 ns > 10 ns
1980s: Alley & Shih, Hong, Ou, Mandel, ...

|ψ⟩ = (r |H⟩ + t |V⟩) (r |V⟩ + t |H⟩) = ...

r^2 |H, V⟩ + t^2 |V, H⟩

"post-select only these events"

1995 Kwiat et al. (Zeilinger group)

New source

Along directions 1, 2:

|ψ⟩ = \frac{1}{\sqrt{2}} (|H_1, V_2⟩ + e^{i\alpha} |V_1, H_2⟩)

(without post-selection etc. !)

"100 standard deviations violation" in < 5min

Coincidence rates > kHz

> 10% detection efficiency
Ideal locality conditions:
Weihs et al. (Zeilinger group, Innsbruck)

Space-like separation
Independent random switching of settings

\[
\frac{400\text{m}}{c} \approx 1.3 \mu\text{s}
\]

\[\Rightarrow \text{choice of setting \& detection need to occur in } < 1.3\mu\text{s}\]

Physical random number generator:

\[\text{LED} \rightarrow \text{D "0" \& "1" \& \sim 10\text{ns}}\]

\[\text{electro-optic modulator rotates polarization by } 0^\circ \text{ or } 45^\circ\]

Setting (voltage) from random number

Total time \(\leq 100\text{ns!}\)

Detectors: collection + detection efficiency 5%

\[\sim 10 \text{ kHz signal counts}\]

\[\sim \text{few } 100\text{Hz dark counts}\]

Time-tag detection events: independent atomic clocks

\[\Rightarrow \text{compare later on computer, extract statistics}\]

\[\rightarrow 30 \text{ std violation!}\]
Ion trap Bell experiment
Windland group 2001

Nearly perfect detection efficiency!

\[ ^{9}\text{Be}^{+} \text{ ions} \]

\[ \begin{align*}
\text{total spin (spin + orbital + nuclear)} & \\
|F=1, m_F=-1\rangle & \\
|\uparrow\rangle & \\
\sim 1 \text{ GHz} & \\
|F=2, m_F=-2\rangle & \\
\end{align*} \]

Control state via "stimulated Raman transition" (two lasers)

\[ 2P_{\nu_2} \]

\[ \left\{ \begin{array}{c}
\text{level scheme}
\\
(\text{hyperfine levels, all part of "ground state" } 2S_{1/2})
\end{array} \right\} \sim eV \]

Control state via "stimulated Raman transition" (two lasers)

\[ \sum_{\text{normal modes in trap}} \]

Coupling to motion during transitions:

laser drive: \[ \hat{H} \sim |e\rangle \langle g| + \hbar c \]

\[ e^{i \hbar \vec{x} \hat{S} x} \]

\[ \approx e^{i \hbar \vec{x} \hat{S} x} (1 + \hbar \vec{x} \hat{S} x + ...) \]

Note: For Raman transition: consider

\[ \Delta \vec{h} = \vec{h}_1 - \vec{h}_2 \]

\[ \vec{h}_1 - \vec{h}_2 \parallel \text{trap axis} \]
Mølmer/Sørensen entangling gate:

\[ |\uparrow\downarrow0\rangle \]

\[ \Omega \]

\[ |\uparrow\uparrow\rangle \]

\[ \text{"two-photon transition"} \]

\[ \Omega_{\text{eff}} = \frac{\Omega_{s}^{2}}{\delta} \]

\[ |\downarrow\rangle \]

\[ |\uparrow\downarrow\rangle \]

\[ \text{detuning} S \]

\[ \omega_{\text{Vib}} \sim 2\pi \cdot 9 \text{ MHz} \]

This is a Raman transition (2 lasers)

\[ \Rightarrow \text{after appropriate time: create} \]

\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle - i |\downarrow\rangle) \]

Detection:

\[ 2P_{3/2} \quad |F=3, m_f=-3\rangle \]

\[ \Rightarrow \text{"resonance fluorescence"} \]

\[ \text{(photon scattering by ion)} \]

\[ |\uparrow\rangle \quad |F=1, m_p=-1\rangle \]

\[ 2S_{1/2} \quad |\uparrow\rangle \]

\[ |F=2, m_p=-2\rangle \]

\[ |\downarrow\rangle \Rightarrow \sim 60 \text{ photons detected} \]

\[ |\uparrow\rangle \Rightarrow \text{dark} \]

\[ \Rightarrow \text{photo-detector detection efficiency not important} \]

Different polarizer settings:

Rotation prior to detection, via Rabi pulse connecting |\uparrow\rangle & |\downarrow\rangle

\[ \Rightarrow \text{"Detection loophole" closed, but "Locality loophole" open} \]