"Quantum optics"	
WS 2016-2017	Worksheet 5

Worksheet 5 contains a discussion of the uncertainty relations for number and electric field operators, as well as for number and phase operators. The paper is devoted to the mass of the photon.

Exercise 11: Quantum fluctuations of the single mode electric field

The single mode electric field is given by

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} \hat{q}(t)\sin(kz)$$

where $\hat{q}(t)$ is the (canonical) position operator.

1) Calculate the expectation values of $\hat{E}_x(z,t)$ and $\hat{E}_x^2(z,t)$ in the number state $|n\rangle$.

2) Evaluate the fluctuations in the electric field given by $\Delta E_x^2 = \left\langle \hat{E}_x^2 \right\rangle - \left\langle \hat{E}_x \right\rangle^2$. Discuss the dependence on n.

3) Compute the commutator $\left[\hat{n}, \hat{E}_x\right]$ and discuss the physical consequences.

4) Show for any Hermitian operators A and B that $\Delta A \Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|$, where $C = [A, B]$. Use the latter to obtain the uncertainty relation for $\hat{n}$ and $\hat{E}_x$.

Exercise 12: Quantum phase

In classical electrodynamic theory, the electric field of a single mode can be written as:

$$\begin{aligned} \vec{E}\left(\vec{r},t\right) &= E_0 \cos\left(\vec{k}\vec{r} - \omega t + \phi\right)\vec{e}_x \\ &= \frac{E_0}{2}\left[e^{i\left(\vec{k}\vec{r}-\omega t+\phi\right)} + e^{-i\left(\vec{k}\vec{r}-\omega t+\phi\right)}\right]\vec{e}_x \end{aligned} \tag{1}$$

where E_0 is the amplitude of the field and ϕ its phase. In order to try to quantize Eq. (1), Dirac introduced the following annihilation and creation operators:

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}} \qquad \hat{a}^\dagger = \sqrt{\hat{n}}\,e^{-i\hat{\phi}}$$

where $\hat{n}$ is the number operator and $\hat{\phi}$ is interpreted as a Hermitian operator for the phase.

1) Using the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, show that $e^{i\hat{\phi}}\hat{n}e^{-i\hat{\phi}} - \hat{n} = 1$.

2) By expanding the exponentials, show that the above equation is satisfied as long as $\left[\hat{n}, \hat{\phi}\right] = i$. Derive the corresponding uncertainty relation (compare Ex. 11.4) for the number and phase observables.

3) Consider the matrix elements of the commutator for arbitrary number states $|m\rangle$ and $|n\rangle$, $\langle m| \left[\hat{n}, \hat{\phi}\right] |n\rangle$ for the case $m = n$.
What does this imply regarding the existence of a Hermitian phase operator?

Exercise 13: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

New Experimental Limit on the Photon Rest Mass with a Rotating Torsion Balance
Jun Luo, Liang-Cheng Tu, Zhong-Kun Hu, and En-Jie Luan
Phys. Rev. Lett. **90**, 081801 (2003)