"Quantum optics"

WS 2016-2017 Worksheet 7

In this worksheet, we discuss the properties of the Wigner function, an important tool in quantum optics, as well as those of squeezed states.

Exercise 17: Wigner function

In classical mechanics, the state of a system can be described with the phase-space distribution $f(x, p, t)$.
$f(x, p, t)dxdp$ gives the probability of finding the system in the phase space element between x and $x+dx$, p and $p+dp$ at time t. In quantum mechanics, such a distribution cannot exist, since x and p cannot be determined simultaneously due to the uncertainty relation. A (pseudo) phase-space probability distribution can however be defined. In 1932, Wigner introduced the function,

$$W(x, p, t) = \frac{1}{2\pi\hbar} \int dy \, \rho\left(x + \frac{y}{2}, x - \frac{y}{2}, t\right) e^{-\frac{ipy}{\hbar}}$$

where $\rho(x, x', t) = \langle x|\hat{\rho}(t)|x'\rangle$ is the density operator of the system in the coordinate representation.

1) Show that the expectation value of an operator $\hat{A}$ is given by $\langle \hat{A}(t)\rangle = \int dxdp \, \tilde{A}(x, p)W(x, p, t)$. Here $\tilde{A}(x, p)$ is the Weyl transform of the operator $\langle \hat{A}\rangle$:

$$\tilde{A}(x, p) = \int dy\langle x - y/2|\hat{A}|x + y/2\rangle e^{\frac{ipy}{\hbar}}$$

   Note: The Weyl transform corresponds to the classical momentum and position dependent observable $A(x, p)$ in the limit $\hbar \to 0$.

2) Show that integrating over the position (momentum) coordinate yields the probability distribution in the momentum (position) coordinate, $W(p) = \langle p|\rho|p\rangle$ ($W(x) = \langle x|\rho|x\rangle$).

3) Calculate the Wigner function for the ground and first excited state (Fock states) of the harmonic oscillator $U(x) = \frac{m\omega^2 x^2}{2}$. Why is $W(x, p)$ not a true probability distribution?

4) Derive the equation governing the time evolution of the Wigner function for the harmonic oscillator starting from the von-Neumann equation. Compare it with the (classical) Liouville equation for $f(x, p, t)$:

$$\left(\frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial x} - U'(x)\frac{\partial}{\partial p}\right) f(x, p, t) = 0$$

Exercise 18: Squeezed state

Consider a squeezed state $|\xi\rangle = S(\xi)|0\rangle$, $\xi = re^{i\theta}$ with squeezing parameter r.

1) Calculate the mean photon number $\langle n\rangle$ and discuss its behavior for small and large r.

2) Evaluate $S(\xi)|0\rangle$ in the limit of weak squeezing. Discuss what kind of Hamiltonian is required to create a squeezed vacuum state.

Exercise 19: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

Observation of Squeezed States Generated by Four-Wave Mixing in an Optical Cavity
R.E. Slusher, L.W. Hollberg, B. Yurke, J.C. Mertz and J. F. Valley
Phys. Rev. Lett. **55**, 2409–2412 (1985)

Measurement of the quantum states of squeezed light
G. Breitenbach, S. Schiller and J. Mlynek
Nature **387**, 472 (1997)