Nonlinear dynamics in quantum synchronization and topological transport

Nichtlineare Dynamik in der Quantensynchronisation und in topologischem Transport

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Abstract

In this thesis we investigate two different aspects of nonlinear dynamical behaviour: First, we study quantum synchronization which is based on nonlinear, so-called limit-cycle oscillators. Second, we investigate the influence of nonlinear interactions on topological transport in bosonic systems.

We begin with the investigation of the synchronization of two optomechanical systems in the presence of quantum noise. With numerical simulations of the full quantum and a semiclassical model, we identify different phase synchronization regimes, although exact phase locking is prevented due to the inevitable quantum noise. Our main results are noise-induced transitions between different synchronization states, as well as noise-induced bistability, i.e., the appearance of a second synchronization state in the quantum regime, even if in the classical, noiseless limit only a single stable synchronization state exists. We give an overview about the classical-to-quantum transition and also compare our findings to synchronization in the presence of classical, thermal noise.

Subsequently, we study the quantum synchronization dynamics of a quantum Van der Pol oscillator coupled to an external periodic driving. This paradigm system for limit-cycle oscillators allows us to derive an effective, analytical quantum model and identify the different dynamical regimes, like overdamped, underdamped, and notably also quantum-coherent phase motion. We explore the distinct signatures of these different dynamical regimes, e.g., in the spectrum or the squeezing. Moreover, we show that in the remarkable regime of quantum-coherent phase dynamics quantum states with negative Wigner densities (like Schrödinger cat states) are preserved for many oscillations of the system. Our effective model gives insight into how to achieve quantum coherence in synchronization dynamics, while numerical simulations of the full model confirm the predicted behaviour.

Finally, we turn to the classical dynamics of a Chern topological insulator in a nonlinear bosonic system. To this end, we study the half Bernevig-Hughes-Zhang model on a square lattice in the presence of a local Kerr nonlinearity. We investigate the stability of a linear edge state in the nonlinear model and present a linear stability analysis to gain insight into the unstable scattering processes enabled by the nonlinearity. In a time evolution of the full nonlinear model, we can show that these processes lead to a spatially periodic modulation of the edge channel on an intermediate timescale. However, eventually we observe that strong nonlinear processes lead to more complicated behaviour and a significant radiation into the bulk.
Zusammenfassung


CHAPTER 1

INTRODUCTION

Nonlinear dynamics generally refers to dynamical behaviour that is described by nonlinear differential equations. Of course, such nonlinearities can be of rather different physical origin and mathematical complexity. They lead to a plethora of phenomena which are not encountered in linear dynamics, e.g., the creation of solitons and limit-cycle oscillations. In this thesis, we study nonlinear behaviour in two rather different situations. First, in the Chs. 2 and 3, we investigate quantum synchronization. Synchronization, even in the classical regime, inherently relies on limit-cycle oscillators and thus requires a nonlinearity. Notably, rather different nonlinearities can lead to such limit cycles: In Ch. 2 they originate from a nonlinear interaction between an optical and a mechanical mode, while in Ch. 3 they arise due to nonlinear dissipation. We introduce the basic concepts of synchronization in the first part of this introductory chapter, in Sec. 1.1, and give an overview about the recent extension of this actually old research field into the quantum regime. In Sec. 1.2 we then introduce topological insulators, to set the stage for our research presented in Ch. 4. In contrast to synchronization, topological transport itself does not rely on any sort of nonlinearity: Our topological insulator model simply consists of a lattice of appropriately coupled linear (harmonic) oscillators. However, we are interested in the interplay of the topological properties of the system with an additional local nonlinearity. While we can investigate both quantum synchronization and nonlinear topological insulators with abstract models, all of our studies were initially motivated by the realizability with optomechanical systems. This is especially emphasized in Ch. 2 where we focus on an optomechanical system as a concrete model for our investigations. Therefore, in Sec. 1.3, we introduce optomechanical systems as a versatile experimental platform and especially point out its nonlinear behaviour (giving rise to limit cycles) and how optomechanical systems can be arranged to form arrays. Finally, after introducing
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In our everyday life, synchronization has several not very well-defined meanings: To “synchronize a watch” usually refers to setting it equal to a reference at a single point in time (relying on its frequency accuracy and stability until a next “synchronization”). Synchronized diving is a sports where several participants try to execute the same moves at the same time - but there is not necessarily something repetitious involved. Even less related, “synchronizing a mobile device” means to create and update a copy of a data set, preferably in real time (i.e. whenever a change is made). Although these examples share at least some features of Huygens’ coupled pendulum clocks, a sound scientific description of “synchronization”
requires a much clearer definition. Throughout this thesis, *synchronization* refers to the adjustment of an oscillator's frequency to another oscillator or some periodic external reference in the presence of a weak coupling. Let us add some remarks to specify and clarify this:

(i) The systems that are to be synchronized need to independently show stable, self-sustained oscillations, even in the absence of coupling. In particular, these systems need to show so-called *limit-cycle oscillations*. This requires nonlinear oscillators and we explain the key features of limit-cycle oscillators in the next section in more detail. Note that also too strong coupling can lead to an effectively indecomposable system.

(ii) Synchronized oscillators eventually oscillate with the same frequency, thus this is also referred to as *frequency locking*. If the frequency of the systems are altered but even after long times do not coincide, one sometimes uses the term *frequency entrainment*. However, note that in literature frequency entrainment is sometimes also used as a synonym for frequency locking.

(iii) If two systems are synchronized, i.e., frequency locked, this implies a fixed phase relation between these systems. This is also called *phase locking*. However, frequency locking only implies that the relative phase is constant. Depending on the actual value of the relative phase (which is not arbitrary if the system is synchronized) we can distinguish different synchronization regimes. Most prominently, one speaks of in-phase or $0$-synchronization if there is (almost) no phase-lag between the oscillators, and of anti-phase synchronization or $\pi$-synchronization if the oscillations are offset by (approximately) half an oscillation, see Fig. 1.1. In Ch. 2 we study the implications of quantum noise for these different synchronization regimes.

(iv) Synchronization can be studied with systems of different coupling geometries, see Fig. 1.2 for an overview. *Mutual* synchronization refers to two (or more) oscillators that synchronize to each other. However, it is also possible to engineer a *one-directional* coupling, where “slave” oscillators synchronize to a “master” system, while the “master” system remains unchanged (because there is no backaction on it). In this scheme, the master system can be viewed as an external drive which is not necessarily a limit-cycle oscillator. If many oscillators are mutually synchronized one distinguishes between local and global coupling, where local can again refer to different types such as, e.g., nearest, or next-nearest neighbour coupling. In this thesis we focus on the conceptually easiest situations of one oscillator coupled to an external drive and the mutual synchronization of two oscillators. For more details on how to deal with oscillator arrays and networks, where one also has to take into account that only a subset of all oscillators might synchronize, see e.g. [11].

Note that related, synchronization-like phenomena are discussed in the literature that might not fulfill all of the above described criteria. However, here we followed the definition of synchronization of, e.g., Ref. [12].
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Figure 1.2: Different synchronization “geometries”. Limit-cycle oscillators can be coupled in different geometries to achieve synchronization. One can divide these into the categories mutual coupling as opposed to one-directional coupling. If the coupling is one-directional, then the external reference can be an arbitrary periodic oscillator, i.e., it is not necessarily a limit-cycle oscillator itself. If mutually coupled systems are build up of many oscillators, one further distinguishes global coupling from local coupling schemes. Here we show nearest neighbour coupling as an example, but many more local coupling schemes exist.

1.1.2 Limit-cycle oscillators

The key element to study synchronization are so-called limit-cycle (LC) oscillations that appear in nonlinear systems. Let us now discuss the features that make them so relevant: The trajectory of a LC oscillator in phase-space is a closed loop that, most importantly, attracts neighbouring trajectories. This implies, that there is a competition between damping and amplification processes that stabilizes the system on the LC trajectory. If we consider a mechanical oscillator as an example, phase space is simply spanned by momentum and position. As shown in Fig. 1.3(a), LCs are not necessarily circularly shaped. However, if they are sufficiently similar to a circle and approximately centered around the origin, then “the phase” of the oscillator can simply be viewed as the polar angle and its oscillations are called quasiharmonic. The notion of the phase can be generalized for arbitrary LCs, however this definition is sufficient for the work presented in this thesis, where we deal with (approximately) circular LCs only.\footnote{For non-circular LCs one defines the phase as linearly growing with time (from 0 to $2\pi$ during one oscillation). See e.g. Ref. [13] for details.} In contrast to its stabilized amplitude, the phase of a LC oscillator is free: A perturbation along the trajectory (thus changing only the phase) remains forever. In the context of synchronization this is of special relevance, because synchronization (phase locking) refers to the process of fixing the free phase of LC oscillators. This also marks the difference of LC oscillators to harmonic oscillators. While a damped and driven harmonic oscillator
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Figure 1.3: *Limit-cycle oscillators.* (a) shows limit cycles in phase space. Limit cycles have closed trajectories that are amplitude-stabilized (green arrows indicate processes driving the system back to this trajectory), while perturbations along the trajectory (blue arrow) remain forever. For an almost circular limit cycle around the origin (blue), the phase of the oscillation is simply the polar angle $\phi$. Limit cycles of more complicated shape (grey) exist and require a generalized definition of the phase. (b) The oscillations of a driven and damped harmonic oscillator as compared to (c) true limit-cycle oscillations. The red arrow indicates a perturbation of the oscillatory motion. While the perturbed harmonic oscillator (dashed red line) returns to its original motion (blue line), the limit-cycle oscillator only relaxes to its original amplitude, while its phase remains perturbed.

leads to amplitude-stabilized trajectories that look rather similar to quasi-harmonic LC oscillations, cf. Figs. 1.3(b) and (c), its phase is already fixed to the external driving force and thus not free to synchronize. This becomes visible, when the oscillator is perturbed. In Fig. 1.3(b) we show the position of the damped and driven harmonic oscillator as a function of time. While the blue line shows its unperturbed trajectory, the dashed red line shows how the same trajectory evolves if it is perturbed by a sudden kick at the time indicated by the red arrow. After this kick, the system returns to its original form of oscillation both in amplitude and phase. In contrast, Fig. 1.3(c) shows quasi-harmonic LC oscillations. There, the perturbed system (dashed red line) returns to the original amplitude, however its phase remains altered as compared to the unperturbed oscillations (blue line).

Mathematically-speaking, a LC oscillator is described by a set of autonomous differential equations, while the drive of the harmonic oscillator leads to an explicit time-dependence of the corresponding differential equations. The phase of a driven harmonic oscillator is thus determined by its relation to the periodic driving and not free. Note that also a non-driven harmonic oscillator does not fulfill the criterions needed to investigate synchronization: In the presence of damping its oscillation simply dies out and, in the absence of both damping
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and driving, its trajectory is not amplitude-stabilized and depends on the initial conditions. Thus, in the context of synchronization, we will only discuss the behaviour of LC oscillators.

1.1.3 Classical synchronization of a Van der Pol oscillator

Due to the widespread appearance of synchronization in different areas of science, general, platform-independent models to study the most fundamental synchronization behaviour are highly desirable. One possibility is to use a phenomenological approach and start with effective phase equations that describe the observed behaviour. The famous Kuramoto model, developed by Yoshiki Kuramoto in 1975 [14], is highly successful in describing synchronization of arrays of globally coupled phase oscillators. It has been extensively studied and applied in various systems, ranging from neural networks to Josephson junctions. A review about the Kuramoto model, its extensions and applications can be found in Ref. [11]. Another approach is to turn to a paradigmatic model for LC oscillators which, to some extent, describes the generic behaviour of all LCs. Especially if we study synchronization of only a few oscillators, this becomes a feasible and insightful approach.

Here, and later on in this thesis, we will follow this second idea and use the so-called Van der Pol (VdP) oscillator as a paradigm for LC oscillators. It is probably the easiest nonlinear extension of the damped harmonic oscillator. In particular, the damped harmonic oscillator is described by

$$\ddot{x} + \omega_0^2 x + \gamma_1 \dot{x} = 0, \quad (1.1)$$

Note the sign-flip of the linear damping term, i.e., $\gamma_1 > 0$ is a linear amplification rate of the VdP oscillator. This disposes energy into the system, without introducing explicit time-dependence. To balance this amplification, a nonlinear damping of rate $\gamma_2$ is introduced. Altogether, this leads to stable self-sustained (LC) oscillations. The trajectories of a VdP oscillator range from quasi-harmonic to almost saw-tooth-like with LC shapes as displayed in Fig. 1.3(a). Although synchronization can be studied with all types of LCs we restrict ourselves to circular LCs, i.e., quasi-harmonic oscillations. In this regime, the VdP oscillator can be approximated by the following equation for a complex amplitude $\beta$:

$$\dot{\beta} = -i\omega_0 \beta + \frac{\gamma_1}{2} \beta - \gamma_2 |\beta|^2 \beta. \quad (1.3)$$

Often, it is useful to express this equation in terms of an amplitude $R$ and phase $\phi$. This is readily obtained by inserting $\beta = R e^{i\phi}$ and splitting Eq. (1.3) into its real and imaginary part, which leads to
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\[ \dot{R} = \frac{\gamma_1}{2} R - \gamma_2 R^3 \]
\[ \dot{\phi} = -\omega_0. \]  

(1.4)

These equations reveal nicely the above stated simplifications of quasi-harmonic oscillations: The polar angle takes the role of the phase and increases linearly with time. The steady-state solution to the amplitude equation \( R_0 = \sqrt{\frac{\gamma_1}{2\gamma_2}} \) determines the radius of the circular LC.

Now we review the synchronization of this version of the VdP oscillator to an external periodic drive of strength \( F \). We will also refer to this driving as force. The driven system is described by the time-dependent differential equation

\[ \dot{\beta} = -i\omega_0 \beta + \frac{\gamma_1}{2} \beta - \gamma_2 |\beta|^2 \beta - F \cos(\omega_d t). \]  

(1.5)

It is convenient to switch into a frame co-rotating with the driving frequency \( \omega_d \), i.e., we use \( \beta \rightarrow \beta e^{i\omega_d t} \) and neglect the fast rotating term proportional to \( e^{2i\omega_d t} \). Then, we finally study solutions to

\[ \dot{\beta} = i\Delta \beta + \frac{\gamma_1}{2} \beta - \gamma_2 |\beta|^2 \beta - F, \]  

(1.6)

where we defined the detuning \( \Delta = \omega_d - \omega_0 \) as the difference between the driving frequency \( \omega_d \) and the intrinsic resonance frequency \( \omega_0 \) of the VdP oscillator. In accordance with the definitions above, if the VdP is synchronized to the external driving, then it eventually oscillates at the drive frequency (instead of \( \omega_0 \)) and establishes a fixed phase relation towards the drive. Hence, in the rotating frame, synchronization corresponds to finding a stable fixed point of the system (if the VdP oscillator effectively oscillates at \( \omega_d \), there is no oscillation visible in the co-rotating frame). In Fig. 1.4(a) we show the region where the VdP oscillator is synchronized (colored region, also called an Arnold tongue), i.e., where a stable fixed point to Eq. (1.6) exists. The color scale shows the corresponding phases which are spread around \( \pi \). The inset on the left indicates the phase convention that we use here and throughout the thesis.

Again, splitting the complex amplitude equation (1.6) into a phase and amplitude equation, we find

\[ \dot{R} = \frac{\gamma_1}{2} R - \gamma_2 R^3 - F \cos(\phi) \]
\[ \dot{\phi} = \Delta + \frac{F}{R} \sin(\phi). \]  

(1.7)

If we neglect the amplitude dynamics and use the steady-state solution \( R_0 \) in the absence of the external drive \( (F = 0) \), the phase equation simplifies to

\[ \dot{\phi} = \Delta + \frac{F}{R_0} \sin(\phi). \]  

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Figure 1.4: Synchronization to an external reference. (a) Arnold tongue, showing the synchronized regime as a function of external force $F$ and detuning $\Delta$. The phase difference to the drive is shown, determined according to the phase convention displayed in the inset. The boundary of the synchronized regime (black lines) are obtained from the Adler equation. The green arrow indicates the force that is used for (b)-(d), the small crosses mark the detunings used in (c) and (d). (b) The observed frequency of the VdP oscillator in the rotating frame as a function of the detuning. The force is fixed to $F/\gamma_1 = 0.02$. Centered around $\Delta = 0$ a synchronization plateau appears, where the system synchronizes to the drive frequency instead of oscillating at its intrinsic frequency (dashed line). (c) and (d) show the potential $U(\phi)$ for different values of detuning. Inside the synchronized regime, (c), the washboard potential features local minima separated by $2\pi$ and the system settles to one of these. The detuning determines the slope. At $\Delta = 0$ (grey) the potential is not tilted and the minima are exactly localized at $\phi = \pi, 3\pi, ...$. At $\Delta/\gamma_1 = 0.005$ the minima are slightly shifted (black). (d) Outside the synchronized regime, e.g. $\Delta/\gamma_1 = 0.01$, no local minima exist. The system simply "slides down" the potential, i.e., the phase is not locked to the external reference but increases with time.
This equation is known as the Adler equation. Notably, it resembles the Kuramoto model if one considers the case of two coupled phase oscillators and rewrites the model for the relative phase. See Sec. 2.2 for a very brief introduction to this special case. The solution to the Adler equation

\[ \sin(\phi) = -\Delta R_0 / F \]  

(1.9)
gives instructive access to the fundamental synchronization behaviour. (i) If there is no initial frequency mismatch, i.e. \( \Delta = 0 \), there are two solutions for \( \phi \) because of the sine-term, i.e. \( \phi = 0 \) and \( \phi = \pi \). However, only one of these solutions is stable and determines the synchronization regime. The stability depends on the sign of the force. Here, if \( F > 0 \), we find that \( \phi = \pi \) is stable and the system is anti-phase synchronized with respect to the drive. Note that due to our model (1.5), a sign flip of the force corresponds only to a phase shift of the periodic drive and thus has no real physical meaning. Although an analysis of Eq. (1.9) predicts \( \phi = 0 \) to be stable for \( F < 0 \), this corresponds to the same anti-phase synchronized regime with respect to the phase-shifted drive. This is fundamentally different in the Kuramoto model, where the force originates from several microscopic parameters and a sign-flip can have real physical meaning, thus allowing for both real in-phase and anti-phase synchronization. (ii) If there is a frequency mismatch between the drive and the VdP oscillator, \( \Delta \neq 0 \), the phase starts to deviate from \( \pi \) (or 0). This can be seen in Fig. 1.4(a). (iii) The border between synchronization and LC oscillations can be explicitly determined from the Adler equation. We use that the sine is bounded and find that synchronization occurs if \( |-\Delta R_0 / F| \leq 1 \). The black lines in Fig. 1.4(a) show these boundaries obtained from the Adler equation, which fit very well to the full numerical solution (color) of Eq. (1.6). (iv) Here we can easily see that synchronization relies on the competition between the frequency mismatch, i.e. the detuning \( \Delta \), and the coupling strength to the external reference, i.e. the force \( F \). It is intuitively clear that synchronization is lost if the detuning becomes too large as compared to the force. The ratio of the linear and nonlinear damping rates that enter via \( R_0 \) only rescales the solution.

In Fig. 1.4(b) we show the actual oscillation frequency of the system that can be observed. We have fixed the force to a value indicated by the small arrow in Fig. 1.4(a) and vary the detuning. The observed frequency follows the detuning (recall that we are in a rotating frame, so this corresponds to oscillations of the VdP oscillator at its intrinsic frequency \( \omega_0 \)) for large detuning. Then it starts to deviate from this (the dashed line shows the frequency of the uncoupled VdP oscillator) and suddenly becomes zero. Over a range of detunings, the observed frequency is constantly 0 (corresponding to \( \omega_d \) in the non-rotating frame). This is the so-called synchronization plateau.

An intuitive picture of synchronization can be gained in terms of envisioning an effective phase particle in a potential, \( \dot{\phi} = -dU(\phi) / d\phi \). From the Adler equation we find the potential

\[ U(\phi) = -\Delta \phi + \frac{F}{R_0} \cos(\phi). \]  

(1.10)
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This is a so-called washboard potential, which we show in Fig. 1.4(c) and (d) for different values of detuning (indicated by the crosses in Fig. 1.4(a)). The Adler equation describes overdamped motion in this potential landscape and synchronization corresponds to the system settling into a local minimum. The system’s invariance under phase changes of $2\pi$ determines the spacing of the minima. The overall slope of the potential is defined by the detuning, while the depth of the minima scale with the force. If the potential becomes too steep, the local minima disappear and the system can no longer synchronize to the external reference, Fig. 1.4(d). Nonetheless, the remaining wavy structure of the potential still alters the intrinsic motion of the VdP oscillator. In the flat regions of the potential the system is almost synchronized and the phase changes slowly with time. This is followed by rapid phase slips at the steep regions in the potential. With increasing detuning $\Delta$ (or decreasing coupling strength $F$), the VdP oscillator returns to its intrinsic oscillation, i.e., the potential transforms into a decreasing linear function.

1.1.4 What changes in the quantum world?

As a final step towards our research, let us now turn to quantum systems and quantum synchronization. We first want to point out some general differences of classical and quantum systems. Most importantly, there is Heisenberg’s uncertainty principle, $\Delta x \Delta p \geq \hbar/2$, where $\hbar$ denotes Planck’s constant and $\Delta x$ and $\Delta p$ are the position and momentum variance. It forbids that a quantum oscillator (actually any quantum system) is localized to a certain position and momentum, i.e., a point in phase space, as the classical oscillators described before. Instead, we visualize a quantum state $\hat{\rho}$ in phase space using the Wigner density

$$W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dy e^{-i p y/\hbar} \langle x + \frac{y}{2} | \hat{\rho} | x - \frac{y}{2} \rangle.$$  \hfill (1.11)

The Wigner density is a quasi-probability distribution from which the probabilities to find the system at a certain position $x/x_{zpf} = (\beta + \beta^*)/\sqrt{2}$ or with a certain momentum $p/p_{zpf} = i(\beta^* - \beta)/\sqrt{2}$ can be obtained by integrating out the other variable. The “size” of the quantum state, i.e., the width of the probability distribution, reflects the amount of noise the system is exposed to. Notably, even in the absence of any noise source, inevitable quantum fluctuations exist, smearing out the quantum state such that the uncertainty principle is obeyed. These so-called zero-point fluctuations (because they occur even at zero temperature) of a quantum harmonic oscillator are $x_{zpf} = \sqrt{\hbar/2m\Omega}$ in position direction (position quadrature) and $p_{zpf} = \sqrt{\hbar m\Omega/2}$ in momentum direction (momentum quadrature). Here, $m$ denotes the mass of the oscillator and $\Omega$ is its resonance frequency.

In order to study quantum synchronization, we need a quantum description of a LC oscillator. Therefore, we introduce a quantum model for the VdP oscillator in the form of a Lindblad Master equation \[15, 16\],

$$\dot{\hat{\rho}} = -i \left[ \omega_0 \hat{b}^\dagger \hat{b}, \hat{\rho} \right] + \gamma_1 \mathcal{D}[\hat{b}^\dagger] \hat{\rho} + \gamma_2 \mathcal{D}[\hat{b}^2] \hat{\rho}. \hfill (1.12)$$

\[10\]
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Figure 1.5: Quantum limit-cycle oscillator. The evolution of the Wigner density of a quantum Van der Pol oscillator, starting from an initial coherent state. The state rotates in phase space and smears out along the limit cycle due to dephasing. After sufficiently long times, a ring-like steady state is reached, where the phase is completely undetermined.

Here, $\hat{b}$ and $\hat{b}^\dagger$ are bosonic creation and annihilation operators, $[\hat{H}, \hat{\rho}] = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}$ denotes the usual commutator, and we set $\hbar = 1$. Linear and nonlinear damping enter via the dissipators where $\mathcal{D}(\hat{O})\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \hat{O}^\dagger\hat{O}\hat{\rho} - \hat{\rho}\hat{O}^\dagger\hat{O}^2/2 - \hat{\rho}\hat{O}\hat{O}^\dagger/2$. Note that the classical equation of motion (1.3) discussed in the previous section is obtained from this quantum model Eq. (1.12) via $\dot{\beta} = \langle \dot{\hat{b}} \rangle = \text{Tr}(\dot{\hat{\rho}}\hat{b})$.

Similarly to its classical analogue, the quantum VdP oscillator features LC oscillations. In Fig. 1.5 we show the time evolution of the Wigner density of the quantum VdP oscillator, resulting in a LC. As discussed above, quantum states are not localized to a single point in phase space so we start from a coherent state centered on the LC. At first, and rather similar to the classical LC oscillation, this state rotates in phase space along the LC, in this case on a circular “trajectory”. However, in contrast to the classical oscillator this rotation in phase space is only a transient behaviour. With time progressing, the state is smeared out more and more along the LC (which here coincides with the meaning “along the classical trajectory”). This increases the phase uncertainty and is called dephasing. The steady state, the actual quantum LC to which we refer to in the rest of this thesis, is represented by a ring-shaped Wigner density. The radius of the ring refers to the mean amplitude of the oscillation and corresponds (approximately) to the radius of the classical circular trajectory. The width of the ring results from amplitude fluctuations, i.e., fluctuation in radial direction. Notably, the phase of this steady state is completely undetermined or, in other words, all phases are equally probable. This is in contrast to the classical situation, where the phase is well-defined at all times and simply increases linearly.

Now that we have described quantum LC oscillators, let us turn to the implications on synchronization, i.e., quantum synchronization. An obvious first question is how does quantum noise affect synchronization? The fundamental laws of quantum mechanics prohibit
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synchronization in the classical sense which would imply localization to a fixed point in a rotating frame. Instead, synchronization leads to a fixed “blob” in phase space, covering a range of phases (but non longer all phases are equally probable), cf. Fig. 3.1. Overall, this requires new tools to measure synchronization in the quantum regime. Furthermore, one can investigate the actual effects of quantum noise: Does it change the transition towards synchronization? Are there any new effects? We study quantum noise effects on synchronized optomechanical systems in Ch. 2. Although we did not mention it before, also classical synchronization can and has been investigated in the presence of, e.g. thermal, noise [12, 17]. While this allows to borrow some tools and expectations from the field of classical synchronization in the presence of noise, it also implies that we are especially interested in quantum effects that are qualitatively different from effects due to classical, e.g. thermal, noise.

Another possibility is to start with a quantum state in the first place. Although we referred to the above displayed LC of the VdP oscillator as a quantum LC, it is not particularly quantum in the sense of non-classical. Applying classical statistical mechanics, probability distributions similar to the displayed Wigner density could be found. However, this is not the case for all kinds of Wigner densities. The Wigner density is called a quasi-probability distribution, because it can become negative. Most prominently, the Wigner density of a Fock state describing a single excitation, or of the famous Schrödinger cat state feature negativities. Notably, also LCs can contain negativities in their Wigner density, making this state a “true” quantum state in the sense that no equivalent classical representation exists. Unfortunately, the quantum VdP oscillator seems not to feature LCs with negativities, but for example optomechanical systems do [18]. It would be interesting to study if such LC oscillators with negativities lead to any new features when synchronized as compared to ordinary, non-negative LC oscillators.

Note that studies targeting these type of questions usually use the term quantum synchronization, although many of them might be better described by synchronization of quantum systems. A quantum system is a system in a parameter regime, where its quantum nature has significant influence on its behaviour. Thus, also the synchronization properties might be modified as compared to synchronization in the classical limit of the same system. However, the modified synchronization behaviour itself does not necessarily show non-classical behaviour. It is thus of special interest to look for genuine quantum signatures of the synchronized steady state, e.g. a negative Wigner density, or in the synchronization dynamics. Discussing the classical synchronization dynamics of the VdP oscillator using the Adler equation, see Sec. 1.1.3, we have argued that synchronization can be related to overdamped motion of the phase in an effective potential. Overdamped motion rules out potential quantum effects, simply because the strong damping dominates the dynamics of the system. Thus the system relaxes too fast to a (potentially) not so interesting steady state to observe any quantum effects in the meantime. The observed dynamics is then necessarily classical-like. Notably, the classical VdP oscillator also features underdamped phase dynamics in a parameter regime where the Adler equation is no longer a good approximation. Underdamped phase motion implies that at least the damping is so weak that it allows to potentially observe quantum
1.1. Synchronization and limit-cycle oscillators

phenomena before relaxing to the steady state. However, other sources of decoherence could still dominate the system. In Ch. 3 we study the synchronization dynamics of the quantum VdP oscillator and show that it actually can become quantum-coherent, i.e., quantum states are preserved for a significant amount of time.

In the next section we give an overview about the state-of-the-art of quantum synchronization. We briefly review which of the above stated questions have been addressed in the literature already and which experimental platforms were proposed to actually observe these effects.

1.1.5 Quantum synchronization

Quantum synchronization has recently attracted increasing theoretical attention. Many publications only appeared during the process of this thesis, including two contributions from ourselves. Here we want to give a brief overview about the state-of-the-art of the field.²

Due to the challenges of treating full quantum models, the present studies can be sorted into roughly two categories depending on the investigated system sizes. Many initial studies investigate the specific synchronization behaviour of one oscillator to an external reference [16, 19–22] or two mutually synchronized systems only [15, 23–31]. This allows for full numerical simulations and some simple analytical models. In contrast, the behaviour of infinitely large arrays is explored within a mean-field treatment in Refs. [15, 32, 33]. Furthermore, in Ref. [34], the classical Kuramoto model is extended to a semi-classical model in order to account for quantum fluctuations. In the spirit of the original Kuramoto model, this allows the treatment of large numbers of oscillators. Finally, Ref. [35] is exceptional in the sense of presenting a study on a simple enough system to also compute some full solution for large arrays, which are compared to mean-field results. Only very recently, first investigation of small networks have been put forward [36, 37].

Various platforms have been studied in the context of quantum synchronization or proposed to be suitable for experimental realizations of models like the VdP oscillator. Certainly, several of these studies were driven by the recent progress in nano- and optomechanical systems approaching the quantum regime. In particular, Refs. [22, 29, 31, 32] are dedicated studies of quantum synchronization in optomechanical systems (see Sec. 1.3.5 for an overview of classical and quantum optomechanical synchronization). In Refs. [16, 25] the authors discuss how optomechanical means can be used to engineer the specific VdP oscillator models that they use to investigate quantum synchronization. However, one could also generally argue that VdP oscillators are a paradigm for LC oscillator in general and thus results can in principle be transferred to the synchronization of optomechanical LCs even without an exact one-to-one mapping. Another very active field of research addresses quantum synchronization of trapped ions or atoms: For example, the motion of single ions/atoms can be driven

²At the time of writing this thesis (early 2017).
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into self-sustained oscillations which can then be synchronized. In Refs. [15, 33] this is associated to VdP oscillator models, while Ref. [28] studies a particular model for phonon lasing. In contrast, the synchronization of two ensembles of atoms studied in Ref. [26] exploits internal atomic transitions. Moreover, early studies consider superconducting qubits synchronizing to a reference oscillator [19, 23]. Although not extensively studied there, superconducting circuits are also mentioned as a promising platform for experimental realizations in Ref. [21, 37]. Furthermore, quantum synchronization is also studied in a system of two coupled nonlinear cavities, i.e., cavities containing a nonlinear medium, [24], in an array of radiating quantum dipoles [35], and for two coupled micromasers [30].

From this summary it has become clear already that the quantum VdP oscillator is a widely studied paradigmatic model for quantum synchronization [15, 16, 20, 21, 25, 27, 33]. Recently, in Ref. [37] a Kerr-nonlinear oscillator was investigated.

To the best of our knowledge, no synchronization experiment has been performed in the quantum regime so far. Nonetheless, first classical synchronization experiments have been recently carried out in some of the proposed systems. In particular, synchronization to an external drive was shown for an optomechanical system [38] and also synchronization between several nano- and optomechanical systems was demonstrated [39–42].

A central theme in quantum synchronization is to find a suitable measure to quantify it. While in a classical, noiseless situation, a sharp transition from no synchronization to synchronization occurs, a similar effect cannot be expected for a finite quantum system due to the inevitable presence of fluctuations. Only in an infinite system, a sharp transition might be expected in the quantum regime and was indeed found using a mean-field approach [15, 32]. For finite systems however, quantum noise blurs the transition and turns it into a crossover where simply the amount of phase slips decreases. Thus, it is a more reasonable approach to try to quantify the “amount” of synchronization instead of defining an artificial border between synchronized and unsynchronized regimes. In Ref. [43] measures for quantum synchronization of continuous variable systems are proposed and upper bounds, e.g. due to the Heisenberg uncertainty principle, derived. These measures were extended in [44] to more generalized meanings of synchronization and applied in a study of synchronization in networks [36]. In Ref. [45], mutual information is proposed as another measure not restricted to continuous variable systems, i.e., that also works in the deep quantum regime. Another measure is defined in Ref. [28]: It is shown how to determine a probability distribution for the relative phase between the synchronized systems directly from the steady-state density matrix of the system. This is a rather intuitive way to visualize the amount of synchronization in a system and also reveals different synchronization regimes. Then, the actual quantum synchronization measure is defined to be basically the peak height of the phase distribution above the flat (not synchronized) distribution. Unfortunately, this no longer contains the information about the synchronization regime and, for example, leads to ambiguous results in

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3Note that (when it comes to mutually coupled oscillators described by Lindblad master equations) there exist two different types of mutual coupling, termed dissipative (coupling via the dissipators) and reactive (the coupling is a term in the Hamiltonian) that can lead to different behaviour.
1.1. Synchronization and limit-cycle oscillators

the presence of double-peaked phase distributions. Such bistable synchronization appears, e.g., in optomechanical systems. To describe such situations, we introduce a specific correlator between the synchronized systems that carries information about the relative phase and allows to distinguish the particular synchronization regimes of interest [29], see Ch. 2. While all of these quantum synchronization measures have been applied in some context, a universal measure applicable to all situations has not yet been identified. For example, in Ref. [30] it is shown that the relative phase distribution measure [28] and the mutual information [45] lead to rather different results. This reflects that the different measures are based on different types of correlations and the general interplay between synchronization and correlations are not well understood yet. Especially entanglement was investigated in several works. While synchronization clearly leads to some sort of correlations, entanglement seems not to be directly tied to it [43]. However, several other synchronization studies report at least some entanglement [23, 24, 27, 30] and in Ref. [33] even an entanglement tongue is discussed as a quantum analogue to the classical Arnold tongue. In Ref. [28] it is discussed how some of the arising correlations could be used to experimentally read-out the synchronization of the coupled ions. In particular, the motion of the ions gives rise to correlations between the motion and an internal “spin” degree of freedom. Thereby also the respective spins of the synchronized ions become correlated and the spin-spin correlation can be used to detect the synchronization dynamics in this specific system.

Despite the lack of a reliable quantification measure, the behaviour of quantum-synchronized systems and particularly the differences from classical synchronized systems have been discussed in several systems. Refs. [16, 25] investigates how the perfect frequency-locking in the classical regime turns into strong frequency entrainment in the quantum system. Also the effects of using LCs of different amplitudes are discussed [25]. Phase locking is studied in [15] and found to be even more robust in a quantum system than in the corresponding (noisy) classical model. A similar effect is mentioned in [30]. In Ref. [29] we find and discuss quantum noise-induced synchronization phenomena. We find that quantum noise drives transitions between different synchronization states and can even induce a bistability if in the classical, noiseless limit only one synchronization state is stable. Multiple resonances in phase and frequency locking (not present in classical synchronization) and even negative Wigner densities are found in Ref. [20]. Notably, the authors argue that a VdP oscillator is not sufficient to lead to negative Wigner densities and thus they include a Kerr-nonlinearity in their model. In Ref. [37], a purely Kerr-nonlinear oscillator is studied and a quantum synchronization blockade predicted. This refers to the rather counterintuitive result that a (small) detuning between oscillators is needed to synchronize them in the deep quantum regime. This is in stark contrast to classical results, where smaller detuning is always desirable. While all of these studies mainly discuss the synchronized steady state, in Ref. [21] we investigate the quantum synchronization dynamics. Most importantly, we show that it is not necessarily overdamped and thus classical-like, but can actually become quantum-coherent.

Note that, although all of these results are descriptions of quantum synchronization behaviour, not all of them represent genuine quantum behaviour, cf. our discussion at the end.
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of the previous section.

The above discussed selection of papers represents the core of the field that is related closely to the research on quantum synchronization that we have performed [21, 29] and which we discuss in the chapters 2 and 3. Various further work on “quantum synchronization” exists, e.g., in the context of an abstract quantum rotator model [46] (which is argued to be relevant for Josephson junctions and cold atoms), quantum tunneling [47], qubit systems [48, 49], superinsulators [50], superconducting devices [51], and linear oscillator systems [52, 53]. However, not all of these studies can be directly understood in the sense of spontaneous synchronization of LC oscillators as we use it in this thesis.

1.2 Topological insulators

1.2.1 Brief overview of topological insulators

As compared to synchronization, the field of topological insulators is remarkably young. It was only realized in the 1980s when the quantum Hall effect was discovered [54], that quantum transport can be protected for topological reasons. Topology is a mathematical field that is concerned with the question which kind of objects can be continuously transformed into each other. This concept of classifying structures into topological equivalence classes was then used to explain the appearance of robust, chiral edge (in 2D) or surface (in 3D) channels in an insulator when an external magnetic field is applied. An insulator’s band structure has a bandgap, i.e., there is a range of energies at which no states are available. Unless sufficient energy is supplied to excite the system above the bandgap, the system remains insulating and no conduction occurs.

At any interface between materials, and of course also at the edge or surface of a material towards air or vacuum, one has to account for boundary effects. Following the idea of topological classification, we can raise the question whether the bulk Hamiltonians at both sides of the interface can be continuously transformed into each other, without closing the bandgap at any point of the transformation. If this is possible, then both materials are called topologically equivalent and the system is (in principle) insulating even across the interface. However, if the bandgap has to be closed in the process this gives rise to conducting states localized along the interface, i.e., at the edge of the material. These topological edge states feature special properties as compared to ordinary conducting edge states, which will be discussed below.

In 1982 Thouless, Kohmoto, Nightingale, and den Nijs discovered a topological invariant [55], sometimes referred to as the TKNN topological invariant, that determines the quantization of the Hall conductance observed in the experiment. This topological invariant can be used to classify systems into topological equivalence classes and was further studied in Ref. [56]. In 1985, it was found that the TKNN invariant is actually related to an already established quantity in math, to the so-called (first) Chern number [57]. This Chern number determines the number of edge states per edge (towards the topologically trivial vacuum)
1.2. Topological insulators

![Diagram](image_url)

**Figure 1.6: Types of topological insulators.** (a) An illustration of an edge state of a Chern insulator. It leads to chiral transport along the edge of the material. (b) An illustration of an edge state of a $Z_2$-topological insulator, which gives rise to helical transport along the edge, i.e., spin-up particles are transported in one, spin-down particles are transported in the other direction.

and also the directionality of the associated transport. It is a remarkable fact that the Chern number can be fully calculated from the bulk properties of the system, i.e., from the infinitely extended lattice description of a material that knows nothing about the actual edges. This so-called *bulk-boundary correspondence* directly implies that, e.g., the actual shape of the edges cannot matter. In contrast, ordinary edge states might occur depending on particular properties of the actual edges but are not guaranteed by the bulk properties as topological edge states.

Before explaining more details about the Chern number and properties of the corresponding edge states in the next subsection, let us finish this brief historical overview. In 1988, Haldane came up with a specific model that leads to chiral topological edge channels without the need of an external magnetic field [58]. This so-called anomalous quantum Hall effect is based on a staggered magnetic field and gives additional insight into the origin of the topological properties. It was understood that not an external magnetic field is crucial, but time-reversal symmetry breaking. This fits nicely to the observed chirality of the edge channels which obviously breaks time-reversal symmetry.

In 2005, Kane and Mele found another class of topological insulators that do not break time-reversal symmetry [59, 60]. Their scheme includes spins and results in helical edge states, i.e., pairs of edge channels where one carries spin-up particles in one direction and the other carries spin-down particles in the opposite direction. Due to the involvement of spins to preserve time-reversal symmetry, this was termed the quantum spin Hall effect. Also this new type of topological insulators can be classified using a topological invariant called $Z_2$. While these insulators have the topological trivial Chern number 0, they are characterized by a nontrivial $Z_2$ number and referred to as $Z_2$-topological insulators. In contrast, the above described topological insulators that break time-reversal symmetry and have a nontrivial Chern number are called *Chern insulators*. In this thesis we will focus on Chern insulators and thus introduce the corresponding concepts in more detail in the following subsection.

Recently, the idea of using temporal-modulation techniques to manipulate a system's properties in order to engineer interesting topological band structures has attracted grow-
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ing attention. Topological (and Chern) insulators created by such means are also referred to as Floquet topological insulators [61].

Although topological insulators were discovered in electronic (and thus fermionic) condensed matter systems the concept has now been transferred to various other fields and bosonic systems as well. For example topological edge states in ultracold atoms [62], and recently even in macroscopic mechanical (phononic) systems [63–65] have been experimentally demonstrated. Also topological states in photonic systems have attracted a lot attention, see, e.g., Ref. [66] for a recent review. Finally note that also optomechanical arrays can serve as topological insulators [67]. We will introduce this in more detail in Sec. 1.3.5.

1.2.2 Chern number and topologically protected edge states

Let us now focus on Chern insulators and discuss how to determine the number of edge states, their directionality, and their properties. Given any (time-independent) bulk Hamiltonian, the bulk band structure is obtained by exploiting the translational symmetry of the underlying periodic lattice structure. In the case of an insulator, this band structure features one or several bandgaps. For each band \( n \) the Chern invariant \( Q_{(n)} \) is given by integrating the respective Berry curvature \( B_{(n)} \) over the Brillouin zone [68],

\[
Q_{(n)} = -\frac{1}{2\pi} \int B_{(n)} \, d\vec{k}. \tag{1.13}
\]

The Berry curvature can be obtained as follows,

\[
B_{(n)} = \vec{\nabla}_k \times i \langle \Psi_n | \vec{\nabla}_k | \Psi_n \rangle, \tag{1.14}
\]

where \( |\Psi_n\rangle \) denotes the respective eigenstate of the bulk Hamiltonian in \( \vec{k} \)-space. Notably, the Chern number is always an integer, i.e., \( Q_{(n)} \in \mathbb{Z} \). Eqs. (1.13) and (1.14) provide a rather technical definition. See, e.g., Ref. [69] for a more practical way of determining the Chern number numerically.

For now let us assume that we know the Chern numbers of all bands in a given band structure, see illustration Fig. 1.7(a). The sum of all Chern numbers has to be zero, i.e., \( \sum_n Q_{(n)} = 0 \) [69]. Now, two band structures are topologically equivalent, if all their Chern numbers coincide pairwise. At an interface of two topologically equivalent systems no edge channels occur, since there is no need to close the bandgap to transform the Hamiltonians into each other. However, if the Chern number has to change across an interface, then edge states appear. The Chern number does not only predict the general appearance of edge states, but actually determines the number of guaranteed edge states and their directionality.

Let us now focus on an edge towards the topologically trivial vacuum. Thus, we can focus on the band structure of the material and the behaviour at the edge is directly given by the its Chern numbers. For example, consider a fictitious band structure with the Chern numbers given in Fig. 1.7(a). Since \( Q^{(1)} = 1 \) (Chern number of the lowest band) we can immediately conclude that there is one topologically protected edge state in the first bandgap. The sign of
1.2. Topological insulators

\[ Q^{(1)} \text{ determines the chirality, i.e., since } Q^{(1)} > 0 \text{ this edge state leads to clockwise transport.} \]

This is illustrated in Fig. 1.7(b), a sketch of a band structure in a material with one edge. The edge state predicted from the Chern number of the bulk band structure shows up in the first bandgap. The slope in Fig. 1.7(b) is associated to the directionality and corresponds to the illustrated movement to the right, i.e., clockwise transport along an upper edge.

In general, the number of topological edge states lying inside a bandgap depends on the sum of Chern numbers of all bands below this particular bandgap. For example, the number of edge states in the second bandgap is given by \( Q^{(1)} + Q^{(2)} \) and the chirality depends on \( Q^{(1)} + Q^{(2)} > 0 \) (clockwise transport) or \( Q^{(1)} + Q^{(2)} < 0 \) (anti-clockwise transport). Consequently, the number and chirality of edge states of the \( n \)th bandgap is determined by \( \sum_{i=1}^{n} Q^{(i)} \). If there exists more than one edge state in a certain bandgap, then the sign determines the directionality of all of these states, i.e., all guaranteed edge states of a Chern insulator in one bandgap move in the same direction. Following this recipe, the number and slope (chirality, direction of movement in the sketch) of all edge states shown in Fig. 1.7(b) can be determined from the Chern numbers in Fig. 1.7(a).

It is important to note that the number of edge states predicted from the Chern number is per edge. For example, if we consider a physical strip instead of an idealized halfplane, then this system has two edges. If the corresponding Chern number is, e.g., +1, this leads to one localized edge state at each edge and two lines cross the bandgap in the corresponding band structure, see Sec. 4.3 for an example.

Let us now shed light on the special properties that make topological edge states so valuable. While, in principle, additional non-topological edge states can appear and disappear

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**Figure 1.7: Chern number and edge states.** (a) Sketch of a fictitious bulk band structure featuring 3 bandgaps. For each band the Chern number \( Q^{(n)} \) can be calculated. (b) Fictitious band structure in a system with one edge. Edge states appear in the bandgaps, their number and directionality (sketched on the right) can be determined from the Chern numbers of all lower lying bands, see main text for details.
Figure 1.8: Robust edge channels. (a) Driving one lattice site with an energy inside the bulk bandgap leads to one-directional (chiral) transport along the edge of the lattice. Since the energy of the drive is within the bulk bandgap no bulk modes but only the edge state can be excited. (b) Notably, the edge channel is robust, e.g., if the edge is damaged and some sites are missing (black), then the transport still follows the more complicated edge. The disappearance of the channel after a finite propagation length is due to dissipation included in this simulation.

depending on the actual shape of the edge, etc., the appearance of topological edge states is guaranteed by the bulk-boundary correspondence (for non-trivial Chern numbers). In this context, one usually speaks of topological protection and robustness of the edge states. Note that the edges can be arbitrarily shaped, including sharp features like additional corners. Furthermore, real systems usually contain some sort of disorder, either by fabrication or by nature. These can be, e.g., lattice impurities, or deviations of the parameters from site to site. All of these effects usually cause scattering. However, because the edge states are located inside a bandgap scattering into the bulk cannot occur, simply because there are no states available at the right energies. Moreover, since Chern insulators feature only chiral edge states, no backscattering can occur. As long as the edge states themselves exist, they reliably transport along the edges with basically no loss into the bulk. The existence of the edge states, however, is what is protected by the topological properties of the system. Changes to the edges of the system cannot change the topological class of the bulk (the Chern number) and thus the edge states remain unharmed. Also mere disorder, etc., in the full system does not directly change the topology, as long as it is of reasonable size, i.e., as long as it does not tear apart the whole system, such that the lattice structure itself can no longer be identified.

In Fig. 1.8 we show an example of this robustness of an edge state on a square lattice (for details of the underlying model see Sec. 4.3). We drive the system on one site with an energy (driving frequency) inside the bulk bandgap, thereby exciting the edge state. In this case, Fig. 1.8(a), transport follows the edge in the anti-clockwise direction without significantly penetrating into the bulk. The overall propagation length along the edge is finite because the simulation also includes dissipation. Figure Fig. 1.8(b) shows a sample with rough edges, i.e.,
marked in black some sites of the lattice are simply missing. The edge state does not disappear and simply follows the more complicated geometry, still without penetrating into the bulk. Exactly this property makes topological insulators interesting for technological developments.

1.3 A promising platform: optomechanical systems

In this section, we introduce optomechanical systems as an experimental platform that can be used to study (quantum) synchronization and (nonlinear) topological insulators. The interaction between the two building blocks of an optomechanical system (optics and mechanics) is intrinsically nonlinear, although the focus in the field has long lasted on the linearized regime. Here, we want to briefly introduce the optomechanical system in general and particularly discuss its nonlinear properties and the multi-mode regime, i.e., its potential to be arranged into large arrays. Coupled optomechanical systems play a central role in Chapter 2 of this thesis, where we specifically study the synchronization of two optomechanical systems.

1.3.1 The interaction between light and mechanical motion

It is a rather surprising fact that light affects mechanical motion. Mostly, because this seems to contradict our everyday life experience, where an object is not visibly pushed back, if, e.g., a laser pointer is directed at it. Nonetheless, this is exactly what happens. A rather simple understanding is possible if we think of light as a stream of photons, each carrying a momentum \(\frac{2\pi\hbar}{\lambda}\) depending on the wavelength \(\lambda\) of the light, with \(\hbar\) Planck’s constant. Then, due to momentum conservation, momentum has to be transferred to the illuminated object. Typically, we cannot see this effect because the transferred momentum is very tiny (around \(10^{-27}\) kg m/s for each single photon in the visible range) and thus many other effects, like friction and heating, dominate the behaviour of the things surrounding us by far.

However, looking into outer space, evidence for the existence of a pressure due to light was recognized already in 1619 by Johannes Kepler [70]: Comet tails are found to always point away from the sun - no matter the motional direction of the comets themselves. Kepler related this observation to a pressure exerted from the sunlight. Although we nowadays know that also solar wind (a stream of charged particles emitted from the sun) contributes to the direction and shape of comet tails [71], this insight of Kepler was groundbreaking. Only centuries later, in 1881, J. C. Maxwell derived that light indeed exerts a pressure along its propagation direction [72]. In the following years, experimentalists struggled heavily to distinguish this radiation pressure from, e.g., thermal effects. In 1901, Lebedew [73] and Nichols and Hull [74, 75] independently succeeded to demonstrate the radiation pressure force experimentally.

Despite those initial struggles to reveal the effects of radiation pressure, nowadays it needs to be taken into account in optical high precision experiments, it might be used to pro-
pel devices in space, and it is the key element for the field of optomechanics. For example, optical interferometers are excellent systems to measure even tiny changes in their relative armlengths. The incredible precisions reached with such systems was recently demonstrated with the first successful detection of a gravitational wave by Advanced LIGO [76]. In order to reach the required precision not only large armlengths of $\sim 4$km were required, but also solid knowledge about all other effects that might displace the freely suspended end mirror of an interferometer arm. One of these effects is radiation pressure. Notably, while the average radiation pressure does not matter since it displaces the end mirrors of both interferometer arms similarly, the effect of radiation pressure fluctuations due to quantum noise turns out to lead to a fundamental limit in precision [77]. However, radiation pressure does not only act as an undesirable and limiting force. For instance, in space where friction becomes negligible and the sun provides a constant and strong light source, several space agencies [78, 79] tried to take advantage of the provided radiation pressure to propel spacecrafts.

To enhance the interaction strength between light and a mechanical object, e.g. a movable mirror, one can simply increase the light intensity. This can be achieved, even without increasing the power of the light source, by introducing a second mirror and creating a cavity. In a cavity, photons bounce between two mirrors for many round trips before they leak into the environment (depending on the quality of the cavity). Because each photon is now reflected from the movable mirror several times, the total amount of momentum transferred to it increases. Notably, in this system, the mechanical motion, i.e., the displacement of the mirror, leads to a backaction on the light because the length of the cavity is changed. This is the archetypical cavity optomechanical system which we discuss in detail in the next section.

### 1.3.2 The archetypical cavity optomechanical system and its mathematical treatment

In Fig. 1.9, we illustrate the prototype of an optomechanical system [80, 81]: A cavity with one static mirror and one mirror attached to a spring, which is thus movable and behaves approximately (for small displacements) like a harmonic oscillator. The static mirror is partially transmitting, thus illumination with laser light allows to populate the cavity mode. Note that the laser driving frequency $\omega_L$ can be detuned from the cavity resonance frequency $\omega_c$. Most light enters the cavity while driving on resonance. The light circulating inside the cavity exerts radiation pressure on the movable mirror, displacing it from its rest position. This changes the cavity length and thereby its resonance frequency $\omega_c(x)$ which depends on the displacement $x$. This, in turn, affects the detuning $\Delta(x) = \omega_L - \omega_c(x)$ between the driving laser and cavity mode, which determines the intensity of the light field insight the cavity and therefore the strength of the radiation pressure force. If the cavity becomes too far detuned the intensity inside the cavity and therefore the radiation pressure force drops. Then the restoring force of the spring takes over and drives the mirror back towards its equilibrium position. This closes the loop of the full optomechanical interaction.

Based on this intuitive description, let us now motivate the optomechanical Hamiltonian
1.3. A promising platform: optomechanical systems

Figure 1.9: The archetypical cavity optomechanical system. A laser drive of frequency $\omega_L$ populates a cavity mode of frequency $\omega_c$. One end-mirror of the cavity is fixed (grey), whereas the other one is attached to a spring and thus free to move (yellow). The movable mirror oscillates with frequency $\Omega$. The radiation pressure of the light displaces the movable mirror, which, in turn, changes the cavity length and thus its resonance frequency.

as the basis for a thorough analysis of the system. Both, the cavity and the mechanical mode are harmonic oscillators which, in the quantized picture, are described using the bosonic annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ and $\hat{b}$, $\hat{b}^\dagger$ respectively. To incorporate the optomechanical interaction, a Taylor expansion of the position-dependent cavity frequency is used: $\omega_c(x) = \omega_c(0) + \frac{\partial \omega_c(x)}{\partial x}|_{x=0} x + ...$. If the mechanical displacements are small, higher order terms of the Taylor series are negligible. The first-order derivative of the frequency is identified as the *bare optomechanical coupling strength* $g_0 = -\frac{\partial \omega_c(x)}{\partial x}|_{x=0} x_{ZPF}$, also called single-photon coupling strength. Here, $x_{ZPF} = \frac{\hbar}{2m\Omega}$ describes the zero-point fluctuations of the mechanical oscillator of mass $m$ and resonance frequency $\Omega$. Using the mechanical position operator $\hat{x} = x_{ZPF} (\hat{b} + \hat{b}^\dagger)$, this leads to the optomechanical Hamiltonian

$$\hat{H} = \hbar \omega_c(0) \hat{a}^\dagger \hat{a} + \hbar \Omega \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a} \left( \hat{b} + \hat{b}^\dagger \right) + \hbar \alpha_L \left( \hat{a} e^{i\omega_L t} + \hat{a}^\dagger e^{-i\omega_L t} \right).$$

It consists of four terms which describe: (i) a single cavity mode, (ii) the mechanical oscillator with resonance frequency $\Omega$, (iii) the optomechanical interaction, and (iv) an optical driving term modeling the laser drive. In agreement with our intuitive understanding of the radiation pressure, the optomechanical interaction term scales with the number of photons in the cavity, i.e., $\hat{a}^\dagger \hat{a}$. Note that the driving term makes this Hamiltonian explicitly time-dependent. It is thus convenient to switch into a frame rotating with the laser frequency. To this end, we apply a unitary transformation to the Hamiltonian, i.e., $\hat{H}_{\text{rot}} = U \hat{H} U^\dagger - i\hbar \partial U^\dagger / \partial t$, with $U = e^{i\omega_L \hat{a}^\dagger \hat{a} t}$. This leads to the very commonly used optomechanical Hamiltonian in the ro-
tating frame,

\[ H_{\text{om}} = -\hbar \Delta \hat{a}^\dagger \hat{a} + \hbar \Omega \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger) - \hbar \alpha L (\hat{a} + \hat{a}^\dagger), \]  

(1.15)

Here, \( \Delta = \omega_L - \omega_c(0) \) denotes the detuning between the laser frequency and the cavity resonance frequency with the movable mirror at its rest position. In the following we omit the reference to the rest position and use \( \omega_c \equiv \omega_c(0) \).

A thorough derivation of the optomechanical Hamiltonian was done by C. K. Law [82], properly quantizing the electric field and finding an even more general expression that simplifies to Hamiltonian (1.15) if the mechanical frequency is small compared to the optical frequency spacing. The general result holds in the nonrelativistic limit (the mirror moves much slower than speed of light) and if the displacement of the mirror is small compared to the cavity length \( L \) and the optical wavelength.

Let us now point out some characteristics of the optomechanical Hamiltonian (1.15): First it is a single-mode Hamiltonian in the sense that it describes the interaction between a single optical and a single mechanical mode. In reality, optical cavities and also mechanical oscillators feature many additional modes. Nonetheless, a restriction to the above applied single-mode description is justified in many (although not all) experiments. The key is to “choose” optical and mechanical modes with frequencies that are sufficiently frequency-separated from other modes. Here, to “choose” means to address, e.g., a specific optical mode by using a driving frequency at or close to the modes resonance frequency; “sufficiently frequency-separated” refers to a comparison between the frequency spacing and the linewidth of the modes. Single-mode optomechanical systems feature already rich phenomena, which we briefly discuss here. Moreover, multi-mode optomechanical systems, especially optomechanical arrays, open up even further possibilities and allow to, e.g., study interesting many-body phenomena, see Sec. 1.3.5.

Second, the optomechanical interaction term in Hamiltonian (1.15) contains combinations of three operators. This leads to nonlinear Heisenberg equations of motion,

\[ \dot{\hat{a}} = i \Delta \hat{a} + ig_0 \hat{a} \left( \hat{b} + \hat{b}^\dagger \right) - i \alpha L \]

\[ \dot{\hat{b}} = -i \Omega \hat{b} + ig_0 \hat{a}^\dagger \hat{a}, \]

(1.16)

that are derived from the optomechanical Hamiltonian via \( \dot{\hat{a}} = \frac{i}{\hbar} [\hat{H}_{\text{om}}, \hat{a}] \) and \( \dot{\hat{b}} = \frac{i}{\hbar} [\hat{H}_{\text{om}}, \hat{b}] \).

Third, in addition to the Hamiltonian description given above, dissipation and fluctuations have to be taken into account to describe the full quantum system accurately. In particular, light leaks out of the cavity at a rate \( \kappa \) which is also referred to as the cavity linewidth or the optical damping. The mechanical oscillator experiences a damping \( \Gamma \), e.g., due to clamping losses. There are several options to incorporate this into the mathematical description of the system: For example one can use a Lindblad master equation of the form \( \dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_j \gamma_j \hat{\rho} \left( \hat{a}_j^\dagger \hat{a}_j - \hat{a}_j \hat{a}_j^\dagger \right) \), as we used it for the quantum Van der Pol oscillator in Sec. 1.1.4 and Ch. 3. Another possibility are quantum jump trajectories where one explicitly includes dissipation in an algorithm by stochastically determining if a loss occurred or not. We use
quantum jump trajectories in Ch. 2 and explain numerical details in the Appendix A.1. On the level of equations of motion one can apply input-output theory [80, 83] to find that $-\frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{a}_{\text{in}}$ has to be added to the optical ($-\frac{\Gamma}{2}\hat{b} - \sqrt{\Gamma}\hat{b}_{\text{in}}$ to the mechanical) Heisenberg equation of motion. Here, the input operators $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{in}}$ represent quantum noise, i.e., they have zero mean $\langle \hat{a}_{\text{in}} \rangle = 0$ and correlators $\langle \hat{a}_{\text{in}}^\dagger(t) \hat{a}_{\text{in}}(t') \rangle = 0$ and $\langle \hat{a}_{\text{in}}(t) \hat{a}_{\text{in}}^\dagger(t') \rangle = \delta(t - t')$ (and for $\hat{b}_{\text{in}}$ respectively). Replacing operators by their expectation values allows to obtain semi-classical equations of motion, which we study in Ch. 2. See Sec. A.2 for numerical details and how to treat the noise terms in the semi-classical limit. In addition to the inevitable quantum noise, all of these methods (master equation, quantum jump trajectories, input-output equations of motion) also easily allow to include finite temperature, i.e., thermal fluctuations. For optical modes in the visible light range it is usually a good approximation to assume a zero-temperature environment, while the mechanical element couples to an environment of finite temperature (which, however, can be lowered by, e.g., executing experiments in a cryogenic environment).

Central features of the optomechanical system are already contained in a linear model. It is common practice to linearize the equations of motion (1.16) around a strong (classical) field $\bar{a}$, i.e. $\hat{a} = \bar{a} + \delta\hat{a}$, where $\delta\hat{a}$ denotes fluctuations around the steady state. In this thesis we focus on the nonlinear regime and only state the most important results in the linear regime. See, e.g., Ref. [80] for an extensive review on cavity optomechanics.

The radiation pressure of the classical field $\bar{a}$ leads to a so-called static shift of the equilibrium position of the movable mirror. This is typically included via an effective detuning $\Delta_{\text{eff}} \approx \Delta + 2g_0^2|\bar{a}|^2/\Omega$. The main (linear) dynamical effects of the optomechanical coupling on the mechanical oscillator are a shift of its intrinsic frequency $\Omega$ (also called optical spring effect) and an optically induced damping, i.e., the mechanical oscillator experiences an effective damping $\Gamma_{\text{tot}} = \Gamma + \Gamma_{\text{opt}}$. If the laser drive is sufficiently weak, the optomechanical damping is given by [80]

$$\Gamma_{\text{opt}} = |\bar{a}|^2 g_0^2 \left( \frac{\kappa}{\kappa^2/4 + (\Delta_{\text{eff}} + \Omega)^2} - \frac{\kappa}{\kappa^2/4 + (\Delta_{\text{eff}} - \Omega)^2} \right).$$  \hspace{1cm} (1.17)

Notably, the sign of $\Gamma_{\text{opt}}$ can be either positive or negative. It acts as an additional damping if $\Delta_{\text{eff}} < 0$, i.e., if the laser drive is effectively red detuned from the cavity resonance frequency. An intuitive understanding can be obtained considering this situation in a scattering picture: An effective red detuning means that the incoming laser drive photons have a smaller energy $\hbar\omega_L$ than photons that match the cavity resonance $\hbar\omega_c$. However, it is possible that a drive photon scatters into a cavity photon (of higher frequency and thus carrying more energy) by annihilating a phonon of energy $\hbar\Omega$ at the same time. Obviously, this works best if the detuning exactly matches the mechanical frequency $\Omega$ such that a phonon exactly fits the energy difference between a laser and a cavity photon. This process is called optomechanical cooling, because the continuous annihilation of phonons effectively cools the mechanical mode.\footnote{Not the total mechanical oscillator is cooled, but only a single mode.} Most importantly, in the resolved sideband regime where $\Omega > \kappa$, this optomechanical cooling is...
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cal cooling can become so strong that the quantum mechanical ground state of the system can be reached \[84, 85\]. This has been a strong drive in the field of optomechanics and was experimentally achieved on different platforms \[86, 87\]. However, in this thesis we are interested in the “opposite” regime. If $\Delta_{\text{eff}} > 0$, i.e., the laser drive is effectively blue detuned, then the optomechanical damping becomes negative, $\Gamma_{\text{opt}} < 0$. In this case, the optomechanical coupling acts as an additional drive of the mechanical oscillator: Laser photons carry too much energy to directly scatter into cavity photons and it is advantageous to deposit energy to the mechanical oscillator by creating new phonons. This amplifies the mechanical motion. Note that the mechanical oscillator is still subject to its intrinsic damping $\Gamma$ and we have to distinguish two cases: Either the total mechanical damping $\Gamma_{\text{tot}}$ remains positive because the intrinsic damping exceeds the optomechanical amplification - then the mechanics is still performing damped oscillations. Or the optomechanical amplification exceeds the intrinsic damping and $\Gamma_{\text{tot}} < 0$. Then, the motion of the oscillator is continuously amplified and the system becomes unstable within the linear framework. This actually marks the onset of self-sustained oscillations, because the nonlinearity of the system finally stabilizes the amplified oscillations to a finite amplitude. Such stable self-sustained oscillations are the basis to study synchronization of optomechanical systems, as done later in this thesis in Ch. 2. It is evident that the linear description only suffices to determine the threshold of self-sustained oscillations ($\Gamma_{\text{opt}} = 0$) and the full nonlinear treatment is required instead to study the regime where the linearized approach only results in instability. We introduce the nonlinear behaviour of the optomechanical system in Sec. 1.3.4 of this chapter.

1.3.3 Experimental realizations and applications

The standard optomechanical system described above is generic in the sense that many different experimental realization comprise the same key elements and are described by the same mathematical model, Eq. (1.15). This variety of implementations gives access to huge parameter ranges and each implementation comes along with its respective advantages and disadvantages. Here we want to give a few examples of different experimental realizations. For a more complete overview we refer to the review article Ref. [80].

Closest to Fig. 1.9 are cavities with one suspended mirror. One example is the LIGO interferometer designed for the observation of gravitational waves mentioned before [76]. Dedicated optomechanical systems in laboratories are usually much smaller, e.g., with mirrors of the size of several micrometers suspended on some sort of cantilever, see e.g. [88–90]. Instead of suspending one end-mirror of the cavity, some experiments place either a membrane [91] or a nanomechanical element, e.g. a nanorod [92], inside a rigid cavity. Notably, such membrane-in-the-middle setups can also feature quadratic optomechanical coupling\(^5\) depending on the exact membrane position inside the cavity with respect to the nodes of the electromagnetic field. Optomechanical systems of rather different geometrical structures are

\(^5\)Coupling of the form $-\hbar g_2 \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger)^2$, i.e., the quadratic term in the Taylor expansion leading to Eq. (1.15). It becomes important if the linear term vanishes.
microdisks, microtoroids, and microspheres. They contain both the optical and mechanical modes simultaneously [39, 93–95]. For example, the light can be guided in a whispering-gallery mode around the structure, while the mechanical mode can be a breathing mode, thus directly influencing the “cavity” length which gives rise to the optomechanical interaction. To decrease the losses that generally arise due to the suspension or clamping of the mechanical element, some optomechanical systems work with levitated mechanical resonators trapped inside the cavity, e.g. dielectric submicron particles [96] or nanospheres [97]. Similarly, in cold-atom optomechanics a cavity mode is coupled to a collective mode of a cloud of trapped atoms [98].

Alternative implementations of the same optomechanical Hamiltonian, are so-called optomechanical crystals, see e.g. [99]. They are based on photonic crystals which are structures made of periodically patterned materials, e.g. silicon, where localized optical modes (“cavities”) are created by introducing local defects to the lattice. In addition, such crystals feature vibrational, i.e., mechanical modes. Engineering crystals where optical and mechanical modes are localized in vicinity to each other gives rise to optomechanical interaction.

Moreover, optomechanical systems can also be realized in the microwave domain. There, the archetypical optomechanical system is realized in a (superconducting) microwave circuit as a $LC$-resonator with a movable capacitor [86, 100].

A recent extension to the field is to replace the commonly used solid-state mechanical oscillators with liquids, coupling an optical mode to an acoustic mode. This gives access to additional degrees of freedom present in liquid dynamics. To this end, one can either fill a cavity with a liquid, e.g. superfluid helium [101], or use a droplet that serves simultaneously as an optical and mechanical resonator [102, 103].

One of the most appealing properties of optomechanical systems is the possibility to obtain quantum control despite its rather macroscopic size (as compared to single atoms or molecules). For example, ground-state cooling has been achieved on different optomechanical platforms, i.e., with microwave circuits [86] and with a nanomechanical oscillator [87]. Many proposals aim to use optomechanical systems to test fundamental quantum mechanical questions, especially addressing the quantum-to-classical transition. The idea is that optomechanical systems allow to create macroscopic superposition states and therefore allow to test different collapse models, see e.g. [104].

More technologically-oriented applications of optomechanical systems can be envisioned as sensors or transducers: For example, optomechanical systems can be used as very sensitive mass [105] and acceleration [106] sensors. Furthermore, transducing between different wavelengths of optical light [107] or even between microwave and optical light [108] has been realized and is of relevance for future quantum communication networks. In addition, the ability to couple optomechanical systems to additional degrees of freedom, e.g. spins or superconducting qubits, allows for potential applications in quantum information processing.
1.3.4 Nonlinear (quantum) optomechanics and optomechanical limit cycles

The intrinsically nonlinear optomechanical interaction gives rise to a variety of interesting effects and plays a central role in this thesis. Considering the nonlinear term in the Hamiltonian (1.15), \( g_0 \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger) \), there are two possibilities to increase its relevance:\(^6\) (i) either the number of photons \( \hat{a}^\dagger \hat{a} \) or (ii) the single-photon optomechanical coupling strength \( g_0 \) has to be enhanced. The first option allows to observe strong nonlinear effects even for small \( g_0 \) and the resulting effects are considered classical. The reason is that for small single-photon coupling \( g_0 \), the interaction between optics and mechanics can be understood as a classical optical field that interacts nonlinearly with the mechanics. In other words, the mechanical oscillator merely feels the impact of individual photons. However, if \( g_0 \) itself becomes large, i.e. larger than the optical linewidth, \( g_0 > \kappa \), even the interaction of a single photon with the mechanical oscillator leads to significant nonlinear effects. This results, for example, in a photon blockade [109], the appearance of photon anti-bunching [110], allows to entangle the optical and mechanical mode [111, 112], and, more generally speaking, allows to create non-Gaussian and quantum states of the mechanics [18, 113, 114].\(^7\)

In state-of-the-art experiments, the regime of single-photon strong coupling, \( g_0 > \kappa \), is not yet reached, but efforts to achieve stronger coupling are made. Furthermore, there are theoretical proposals to increase the effective nonlinear interaction in the context of multi-mode optomechanical systems [115–117].

Meanwhile, classical effects of the nonlinear interaction have already been investigated both theoretically and experimentally. For example, the classical nonlinearity leads to optomechanical bistability [118, 119], i.e., parameter regimes where two solutions of different optical (and mechanical) occupation are stable. In the blue detuned regime, the nonlinear interaction counteracts the optomechanical amplification such that the mechanics settles into self-sustained oscillations, i.e., the optomechanical system behaves like a limit-cycle (LC) oscillator [120–122]. It also gives rise to dynamical multistability [123], i.e., an intriguing attractor diagram of LCs of different amplitude, which has also been observed experimentally [124, 125]. Going beyond LC motion, even chaotic behaviour has been observed [122, 126]. Notably, these effects of the classical nonlinearity also exist in the quantum regime, with the quantum nonlinearity possibly leading to genuine signatures. In particular, LC motion in the quantum regime was explored theoretically in Refs. [18, 127, 128], notably even identifying LCs with negative Wigner densities [18, 128]. Another recent study in the quantum regime sheds light on the transition towards chaos [129].

After this brief overview, let us turn to a more detailed discussion of optomechanical LC motion, which is the basis for our research on optomechanical synchronization in Ch. 2. As mentioned before, blue detuned driving of an optomechanical system leads to amplification:

\(^6\) If we exclude an increased mechanical displacement \( \hat{b} + \hat{b}^\dagger \), which would harm the approximations in the derivation.

\(^7\) Note that this is not the case in the linearized optomechanical model, because the bilinear Hamiltonian preserves Gaussian states. Only the full nonlinear Hamiltonian can lead to non-Gaussian results (unless a quantum source is used to drive the system).
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Figure 1.10: **Threshold to self-sustained oscillations and optomechanical quantum limit cycles.** (a) Below threshold, if the optical driving is too weak, any excitation of the mechanical resonator is rapidly damped out. (b) Above threshold, the optomechanical interaction amplifies the mechanical motion and drives the mechanics into stable self-sustained, almost sinusoidal, oscillations with a fixed amplitude. (c) shows the maximal amplitude in position and momentum as a function of the driving strength $\alpha_L$. The system enters the regime of self-sustained oscillations via a Hopf bifurcation, here close to $\alpha_L/\Omega = 0.2$, where the amplitude becomes nonzero followed by a fast increase. This amplitude is obtained from the classical equations of motion. Below we show the actual quantum limit cycles corresponding to several different driving strengths. (d) Well above threshold, a clear optomechanical limit cycle has emerged. It is slightly displaced from the origin due to the static optomechanical shift and not perfectly symmetrically distributed weight. Parameters: $\Gamma/\Omega = 0.015$, $\kappa/\Omega = 0.3$, $g_0/\Omega = 0.3$, and $\Delta/\Omega = 0.15$. 

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**Figure 1.10**
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of the mechanical motion. However, below threshold, the intrinsic mechanical damping \( \Gamma \) still dominates the behaviour and every excitation of the mechanics is rapidly damped out. An example of this behaviour is shown in Fig. 1.10(a), obtained by simulating the classical optomechanical equations of motion.\(^8\) If the laser driving strength \( \alpha_L \) is further increased, this behaviour changes suddenly. Above threshold, the mechanical motion is amplified, see Fig. 1.10(b). As a function of time, the amplitude of the oscillation first grows (as also predicted within the linearized optomechanical description with a negative total damping \( \Gamma_{\text{tot}} \)), but settles to a stable, finite amplitude due to the full nonlinear interaction. In its steady state, the system performs so-called self-sustained oscillations that experience no effective damping over time. In fact, the mechanical oscillator is still subject to its intrinsic damping \( \Gamma \), but this is fully balanced by the amplification via the optomechanical interaction. It is a very notable fact that the mechanics, by itself only a damped harmonic oscillator, becomes a LC oscillator via the nonlinear optomechanical interaction. Fig. 1.10(c) shows the emergence of self-sustained oscillations as a function of the driving strength, also obtained from the classical equations of motion. Above threshold, in Fig. 1.10(c) close to \( \alpha_L/\Omega = 2 \), the steady-state amplitude of the oscillations suddenly becomes nonzero and rises. The underlying process can be understood in terms of a fixed point, corresponding to a non-oscillatory steady state, that is losing stability and thereby gives rise to LC motion.\(^9\) In nonlinear dynamics, this transition is known as Hopf bifurcation. Figure 1.10(c) also shows Wigner densities of the corresponding mechanical quantum states for different driving strengths. These Wigner densities are obtained by numerically solving the Lindblad master equation for the full optomechanical system and tracing out the optical states. At zero driving, the mechanical oscillator is found in its ground state (we assume a zero temperature environment) and the size of the corresponding Wigner density is determined by the zero-point motion, i.e., quantum fluctuations. With increasing driving, but still below threshold, the size of the Wigner density, i.e. the uncertainty, increases because the mechanical oscillator is now also subject to the laser shot-noise via the optomechanical interaction. Above the threshold, where also classically LC oscillations arise, the Wigner density transforms into a ring shape. At first, however, the LC is so small that the broadening effect of the quantum noise leads to significantly non-zero values of the Wigner density even in the center of the circle. Nevertheless, at higher driving strength, a “clearer” LC forms where the Wigner density inside the circle becomes vanishing. The radius of the LC increases with the driving strength \( \alpha_L \), corresponding to a larger amplitude of the oscillations. In Fig. 1.10(d), we show one LC for closer inspection. Optomechanical LCs are almost circular, however, as opposed to the LCs of the Van der Pol (VdP) oscillator discussed in Sec. 1.1.4, they are not necessarily perfectly symmetric. In Fig. 1.10(d), we see that this LC has slightly more weight at the top right corner. This effect vanishes in the limit where the oscillator performs quasi-free oscillations, i.e., where the damping and the optical driving both become small, \( \Gamma \to 0 \) and \( \alpha_L \to 0 \), while the amplitude of the LC is kept fixed via \( \Gamma / |\alpha_L|^2 = \text{const} \). Furthermore, a very close look reveals that the LC is not per-

\(^8\)The classical equations of motion are obtained by taking expectation values of Eqs. (1.16).

\(^9\)The terms self-sustained oscillations and LC oscillations are used interchangeably.
fectly centered around the origin. This is due to the above mentioned static displacement of the optomechanical system: The mechanical system does not oscillate around the original equilibrium position (the origin), but around a shifted, optomechanically induced new equilibrium position. Consequently, for larger driving strengths this shift increases.

These two properties, the circular shape and the displacement, are important in the context of our later study of synchronization, where we need to define the phase of the mechanical LC oscillation. Similar to the circularly shaped VdP LCs it is easiest to use the polar angle as the phase. The asymmetries described above cause an error in this approach. It is rather simple to determine at least the static shift and correct for it. However, due to other computational constraints we have to restrict our analysis to rather small LCs anyways. The particular LC that we investigate in Ch. 2 is shown in Fig. 1.10(c) for $\alpha_L/\Omega = 0.3$. There the displacement from the origin is very small and can be safely neglected. Thus, the definition of the phase as the polar angle is sufficient for our study of optomechanical synchronization.

### 1.3.5 Multi-mode optomechanics and optomechanical arrays

As the field of optomechanics developed, increasing attention has been drawn towards systems where more than a single optical and a single mechanical mode are involved. A growing number of theoretical proposals take advantage of the possibilities of *multi-mode* optomechanical systems to envision promising applications, such as quantum state transfer [130] between two cavity modes. This is particularly useful if the involved cavity modes reside in different frequency domains, e.g., one microwave and one optical mode, such that the coupling to the shared mechanical oscillator makes this system a quantum transducer. Such three-mode optomechanical systems, consisting of two cavity modes coupled to a shared mechanical oscillator, cf. Fig. 1.11(b), have also been proposed to enhance the quantum non-linearity [115, 117], to generate multi-partite entanglement [131, 132], to investigate Landau-Zener dynamics [133, 134], or more recently to investigate heat transport [135] and to achieve sub-Poissonian phonon lasing [136].

On the other hand, also setups with several mechanical resonators coupled to a global optical mode, cf. Fig. 1.11(c), have been theoretically explored. For example, schemes for a back-action evading measurement of the collective mechanical quadrature have been developed [137], or to generate two-mode squeezing between the two mechanical oscillators [138].

First multi-mode optomechanical experiments demonstrated, e.g., wavelength conversion [107, 108, 139] and phonon lasing [140], without yet reaching the theoretically discussed quantum-coherent conversion or observing sub-Poissonian statistics.

Consequently following the idea to increase the number of modes leads us to *optomechanical arrays*. In this context, one optomechanical system is simply one *site* in a periodic lattice structure. Coupling between different sites can be either of mechanical or optical nature, realized for example via phonon and/or photon hopping between nearest neighbours.
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Figure 1.11: Optomechanical arrays. (a) We depict a single optomechanical system by a combination of two colored circles, where the blue circle denotes the optical and the yellow circle the mechanical mode, and their placement on top of each other implies onsite optomechanical interaction. (b) and (c) show examples of multi-mode systems, where in (b) two optical modes couple to the same mechanical mode, while in (c) two mechanical modes share one optical mode. (d) and (e) show example geometries of periodic arrays of optomechanical systems. Coupling between different sites occurs, e.g., via optical (blue lines) and/or mechanical (yellow lines) hopping.

Such an optomechanical array would then be described by a Hamiltonian of the following form,

\[ \hat{H}_{\text{array}} = \sum_j \hat{H}_{\text{om},j} - \frac{J}{z} \sum_{\langle i,j \rangle} \left( \hat{a}_i^{\dagger} \hat{a}_j \right) - \frac{K}{z} \sum_{\langle i,j \rangle} \left( \hat{b}_i^{\dagger} \hat{b}_j \right), \]  

(1.18)

where \( z \) is the coordination number, \( \langle i,j \rangle \) denotes summation over nearest neighbours, and \( \hat{H}_{\text{om},j} \) refers to the standard optomechanical Hamiltonian (1.15) for each site \( j \). Here, \( J \) denotes the photon hopping rate and \( K \) the phonon hopping. Note that this scheme can be generalized even further, allowing, e.g., for next-nearest neighbour coupling or for different parameters of the optomechanical systems at different sites. This would be an even more realistic model, since fabrication inaccuracies typically lead to slightly varying values of the intrinsic frequencies from site to site.

Depending on the actual arrangement of sites and the coupling geometry, different lattice structures such as a square lattice, Fig. 1.11(d), or a triangular lattice, Fig. 1.11(e), can be engineered. This opens up many new possibilities to study many-body dynamics and is of potential interest for quantum simulation purposes. Of course, one can also envision non-periodic network structures or combinations of arrays with "global" couplings - which turn out to be closer to the first experimental realizations.

Theoretically, optomechanical arrays were first brought up to study (classical) synchronization of optomechanical limit-cycle oscillators [141] and in an approach to slow and stop...
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This triggered a large variety of theoretical studies exploring the properties and possibilities of optomechanical array structures: They were studied in the context of reservoir engineering [143], quantum information processing [144], and quantum many-body dynamics [32]. Even topological phases were predicted for optomechanical arrays [67]. Furthermore, photon propagation in a 1D lattice [145], Dirac physics [146], the creation of artificial magnetic fields for photons [147], Anderson localization [148], and dynamical gauge fields [149] were discussed. Several studies addressed in particular multi-membrane systems [150], where an array of mechanical membranes is located in a single cavity. In particular, long-range interactions and dynamics were investigated [151, 152] and it was pointed out that the optomechanical coupling can be enhanced in such systems [116].

Let us now turn to the experimental realizations of optomechanical arrays beyond the few mode setups mentioned above. From the very beginning, Refs. [141, 142], optomechanical crystal structures [99, 153–157] were proposed as promising candidates to produce even large scale arrays. In such systems, by patterning the crystals appropriately, it is possible to fabricate optomechanical systems in the vicinity of each other. This enables tunneling (optical and/or mechanical) between the systems, thereby effectively coupling the sites.

So far, several small optomechanical arrays were realized to experimentally study classical synchronization. This has been successfully achieved with micro-disk resonators [39, 42, 158] and nanomechanical resonators coupled to an optical racetrack [40]. Indeed, the largest optomechanical array reported so far consists of seven disk resonators [42]. Note that in Ref. [40] a shared optical mode is used to synchronize two mechanical oscillators and this setup is more accurately noted (following our own convention that we applied above) as a multi-mode optomechanical system instead of an actual array. In contrast, the micro-disk experiments involve clearly separate optomechanical systems. However, their arrangement and coupling schemes do not implement a periodic lattice as envisioned in Fig. 1.11(d) or (e). Fabricating such lattices of similar optomechanical systems to a high accuracy remains an outstanding challenge on its own. Nonetheless, already small arrays, comprising, e.g., only two sites allow interesting studies: In addition to the above mentioned synchronization experiments, non-reciprocity in optomechanical systems was demonstrated very recently, both with an optomechanical crystal circuit and microwave circuits [159–161].

Before we reach the end of this overview on multi-mode optomechanics and arrays, we want to briefly highlight the topics of this thesis and thus give some additional details on optomechanical synchronization and an optomechanical realization of a Chern topological insulator.

Optomechanical synchronization is based on optomechanical systems that are driven into limit-cycle oscillations, cf. Sec. 1.3.4, and weakly coupled to each other. This was first discussed in Ref. [141] in the classical regime. Both synchronization to an external reference and mutual synchronization between two (and more) optomechanical systems were discussed. To this end, an effective classical model, now called Hopf-Kuramoto model, was derived. It is based on phenomenological parameters that, in principle, can be determined from the microscopic parameters of the optomechanical system. In Ref. [32] it was found that
the amplitude dependence of the frequency can have significant influence (which was ne-
glected in [141]) and thus another term was added to the Hopf-Kuramoto model. A complete
and comprehensive derivation can be found in the Supplementary Material of Ref. [162]. For
our purpose, the relevance of this model is that it gives us an insight into the different types
of optomechanical synchronization regimes that exist. We explain the case of two coupled
optomechanical systems in Sec. 2.2, as a classical reference for our optomechanical quan-
tum synchronization study. Remaining in the classical regime, the Hopf-Kuramoto model
allows to study even large arrays: As an extension to synchronization, i.e. strict phase lock-
ing, phase pattern formation was investigated [162] and, including classical noise, relations
to the Kardar-Parisi-Zhang model were explored [163]. Independent of the Hopf-Kuramoto
model, Ref. [164] investigates classical synchronization in a different coupling geometry, us-
ing one global optical mode instead of a lattice of individual optomechanical systems. A first
discussion of optomechanical quantum synchronization was presented in Ref. [32]. In par-
ticular, the effects of quantum noise were addressed and a mean-field approach allowed to
observe a sharp transition towards synchronization in the limit of infinitely extended arrays.
Quantum synchronization measures and fundamental bounds were discussed in [43], explic-
itly treating optomechanical systems as an example. This was extended in [44]. In [29], see
also Ch. 2, we present a detailed, mainly numerical study of quantum synchronization of two
coupled optomechanical systems. We show that quantum noise (but also classical noise to
some extent) drives transitions between different synchronization states and can even in-
duce the appearance of an additional synchronization state. Very recently, an in-depth study
of optomechanical synchronization investigated the difference between coupling to an ex-
ternal optical drive or directly to an external mechanical drive, also discussing their results in
the quantum regime [22]. In Ref. [31], the quantum synchronization behaviour of a specific
optomechanical setup, two membranes in a Fabry-Pérot cavity, was studied.

Since it is rather challenging to investigate the full quantum model of optomechanical
synchronization, it is advantageous to find a simplified model. In Ref. [16] it is explained how
optomechanical systems can be manipulated to behave like a quantum Van der Pol (VdP)
ocillator. In this thesis we introduce the classical VdP oscillator in Sec. 1.1.3; the quantum
version first appears in Sec. 1.1.4, but is studied extensively in Ch. 3. Further studies based
on this conceptually easier model explored different aspects of quantum synchronization
[16, 20, 25], including our own work [21], see also Ch. 3. It is an additional advantage of the
VdP oscillator that it is not only a suitable model for optomechanical systems, but for LC
oscillators in general. Thus it provides a basis for even more fundamental insights. There
are several studies about quantum synchronization of trapped ions based on the same VdP
oscillator model [15, 33].

Experimentally, optomechanical synchronization has been demonstrated in the above
mentioned small array systems [39, 40, 42], but also towards an external reference [38]. Not-
tably, in Ref. [42] a reduction of the phase noise was observed in the synchronized regime.

A topological insulator, more precisely a Chern insulator, based on optomechanical ar-
rays was first proposed in Ref. [67]. Inspired by optomechanical crystal structures, an analy-
sis of a Kagome lattice geometry was presented. Note that the general optomechanical array Hamiltonian (1.18) itself is not sufficient to turn the system into a topological Chern insulator since it does not break time-reversal symmetry. The idea presented in Ref. [67] is to exploit the optical drive of the optomechanical systems to imprint site-dependent phases on the optomechanical coupling. In particular, the three optomechanical sites per unit cell in the Kagome lattice are subject to a driving laser with a specific, spatially varied phase field. This breaks time-reversal symmetry and turns the system into a Chern insulator. For weak coupling, this can be understood as using the optics and the optomechanical interaction to engineer an effective non-trivial mechanical coupling, such that actually only the mechanical lattice becomes a Chern insulator. Moreover, in the strong-coupling regime the photonic and phononic bands hybridize. In this regime, topological phases can be observed that are no longer of either phononic or photonic nature only. Both situations lead to topologically protected edge states, either for the mechanics or the arising photon-phonon polaritons. A rich diagram of several distinct topological phases (characterized by different sets of the Chern numbers) is observed. A specific advantage of using an optomechanical system is the ability to explore these topological phases by in situ tuning of the laser frequency and/or amplitude. Furthermore, topologically protected phonon transport along the edge of a finite system can be easily induced with an optical probe laser (in addition to the pump that carries the phase pattern to create the non-trivial topology). Due to the optomechanical interaction, this mechanical edge state can also be observed by optical means. Optomechanical arrays were also suggested as possible implementation in further studies of topological properties [165, 166].

1.4 Outline of this thesis

After introducing the basic terms of the research field covered in this thesis, the rest of the thesis is structured as follows.

First, in Ch. 2 we present our research on the synchronization of two optomechanical systems in the quantum regime. This is a mainly numerical study dealing with the full, non-linear quantum model to determine the true quantum solution and applying a semi-classical approach to study the classical-to-quantum crossover. We identify different synchronization regimes in the presence of quantum noise. Most interestingly, we find a regime of "mixed" synchronization where fluctuations drive transitions between classical synchronization states. We focus on this regime and investigate the corresponding classical-to-quantum crossover. This reveals that quantum noise cannot only drive transitions between the states of a classically bistable system, but the noise can also induce bistability. Moreover, we compare the effects of quantum and thermal noise, showing that thermal noise can lead to qualitatively similar results.

Then, in Ch. 3, we present results on quantum synchronization obtained from the more general quantum Van der Pol oscillator model. We study the conceptually easiest situation where a limit-cycle oscillator is synchronized to an external drive. In contrast to our optomechanical quantum synchronization study, we aim for effects even beyond the modifications

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1. Introduction

of synchronization in a quantum system. We identify quantum properties of the synchro-
nization dynamics itself, i.e., we discover quantum-coherent phase dynamics. To this end, we derive an effective quantum model, which allows us to quantify the relevant decoherence and damping rates, and verify our results with numerical simulations of the full system. We give instructions how to find long coherence times (as compared to the dynamics of the system). Furthermore, we briefly discuss some possible experimental realizations, i.e., physical systems that behave like a Van der Pol oscillator in some parameter regime, and argue that the required parameters to observe quantum coherence are realistic. Following the discussion of our main results, we give further insights into the involved models: We explain additional details about the corresponding classical synchronization, we show a comparison between our analytical effective quantum model and the full numerical outcome, and we discuss additional details about the squeezing, the long-time evolution of the system, and the frequency spectrum. This concludes the first part of the thesis that addresses different aspects of quantum synchronization.

We move on to discuss large arrays of coupled oscillators where, beyond synchronization, various other phenomena can arise. In particular, in Ch. 4 we investigate the classical dynamics of a bosonic Chern topological insulator model in the presence of a local nonlinearity, which is quartic on the Hamiltonian level. First, we give an overview about the current status of the young field of nonlinear topological insulators and introduce the linear half Bernevig-Hughes-Zhang model, which is the basis for our study. Then, we investigate the stability of a linear edge state in the nonlinear system, and explore its time evolution. In particular, we present a linear stability analysis and identify the nonlinear scattering processes that lead to instability. Furthermore, we show numerical results of the full time evolution, where we first identify periodic modifications of the linear edge state that can be understood with the help of the stability analysis. At later times of the time evolution, the nonlinear effects become too strong and the edge state eventually decays into the bulk.

We conclude the main part of this thesis in Ch. 5 with a general outlook, discussing the prospects of both quantum synchronization and nonlinear topological insulators. Additional details on the numerical methods used throughout this thesis can be found in the Appendix A.
1.4. Outline of this thesis
In this chapter we present our results that were published in this form in

- Talitha Weiss, Andreas Kronwald, and Florian Marquardt
  Noise-induced transitions in optomechanical synchronization
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Some minor formulations have been changed to assure consistency within this thesis. We also updated a reference from the preprint to the published version.

We study how quantum and thermal noise affects synchronization of two optomechanical limit-cycle oscillators. Classically, in the absence of noise, optomechanical systems tend to synchronize either in-phase or anti-phase. Taking into account the fundamental quantum noise, we find a regime where fluctuations drive transitions between these classical synchronization states. We investigate how this “mixed” synchronization regime emerges from the noiseless system by studying the classical-to-quantum crossover and we show how the time scales of the transitions vary with the effective noise strength. In addition, we compare the effects of thermal noise to the effects of quantum noise.
2. (Quantum) Noise-induced phenomena in optomechanical synchronization

2.1 Introduction

The field of cavity optomechanics deals with systems where light in an optical cavity couples to mechanical motion [80]. Recent experimental progress allows to move on from the investigation of a single optomechanical system to several coupled optomechanical systems. There are already first experiments that involve a few mechanical and optical modes [107, 139, 140, 167], exploiting them for wavelength conversion, phonon lasing and efficient cooling. Such few-mode optomechanical setups have been the subject of an increasing number of theoretical proposals, on topics such as efficient state transfer [130], two-mode squeezing [138], back-action evading measurements [137], entanglement [131, 132, 168] or Landau-Zener dynamics [133, 134].

Larger arrays may be implemented using a variety of settings, such as coupled disks [39] (Fig. 2.1(b)) or optomechanical crystal structures [99, 153–157] (Fig. 2.1(d)). Optomechanical arrays have also attracted attention from a theoretical point of view. They have been studied in the context of slowing light [142], Dirac physics [146], reservoir engineering [143], artificial magnetic fields for photons [147], heat transport [135], and topological phases of sound and light [67]. Furthermore, multi-membrane systems [116, 150–152] were studied theoretically, considering for instance long-range interactions and dynamics.

Most notably, optomechanical arrays provide a platform to study synchronization of mechanical oscillators. This was initially pointed out in Ref. [141]. Synchronization is a well known phenomenon in many different branches of science [12] and typically arises whenever there are stable limit-cycle oscillations. However, we note that synchronization-like phenomena have been recently studied also in the context of linear oscillators dissipating into a common bath [52]. Optomechanical systems exhibit a Hopf bifurcation and can be optically driven into mechanical limit-cycle oscillations [120, 121, 123, 124]. These self-oscillations have also been analyzed theoretically in the quantum regime [127, 128]. The theoretical description of the synchronization dynamics in optomechanical arrays has initially focussed on the classical regime [141, 164]. More recent insights into this regime include the pattern formation of the mechanical phase field in larger optomechanical arrays [162]. In the quantum regime, it was found [32] that quantum noise can drive a sharp nonequilibrium transition towards an unsynchronized state in an extended array, even for optomechanical systems with identical frequencies. Further general insights into quantum synchronization were gained in a model system of one van-der-Pol oscillator coupled to an external drive [16] or two coupled van-der-Pol oscillators [25], which can serve as a rough approximation to an actual optomechanical system. A number of more recent works have explored quantum synchronization on the more conceptual level [34], as well as in various other physical systems, such as e.g. trapped atoms and ions [15, 26, 28, 33], qubits [23, 48, 49], and superconducting devices [51]. The relation of synchronization and correlations in the quantum-to-classical transition was studied in a system of coupled cavities containing a nonlinearity [24]. Notably, it is still challenging to define a good measure for quantum synchronization [28, 43, 45].

Only recently synchronization of two nanomechanical oscillators in the classical regime
was demonstrated experimentally. This has been achieved using optomechanical systems with coupled micro-disks [39] (Fig. 2.1(b)), as well as in an experiment involving an optical racetrack cavity coupled to two mechanical oscillators [40] (Fig. 2.1(c)), and also in a setup using nanoelectromechanical systems [41]. In a recent first step towards larger arrays, up to seven optically coupled micro-disks were used to demonstrate the expected phase noise reduction due to synchronization [42]. This $1/N$ phase noise reduction with the number of coupled systems $N$ is considered to be one of the main prospects of synchronized nanomechanical arrays. Indeed, recognized from the very beginning with the synchronization of pendulum clocks [1], synchronization has the potential to improve time-keeping and frequency stability. Examples where different types of synchronization have been applied or suggested for application are for frequency stabilization of high power lasers by coupling to a more stable, low power laser [6] and for secure communication in connection with chaos [8]. For a more complete overview and also the many applications to biology see e.g. [9, 10, 12].

In this work we study the effects of quantum and thermal noise on the synchronization of two optomechanical systems. We focus on a bistable synchronization regime that either exists already in the absence of noise, or is induced by it. Bistabilities in quantum systems have been investigated before [169–171] and noise-induced bistabilities are known in several other systems in biology and chemistry [172–174], as well as in physics [175–177]. We find and discuss noise-induced bistable behaviour now in the context of optomechanical (quantum) synchronization.

This chapter is organized as follows: We begin with a brief review of classical synchronization in the absence of noise, Sec. 2.2. Then, we introduce our model in Sec. 2.3, explain our methods in Sec. 2.4, and state our main results in Sec. 2.5. We note that both quantum and thermal noise lead to similar effects, but start out with the investigation of quantum noise effects. Therefore, in Sec. 2.6, we simulate the full quantum behaviour of the system, in contrast to the previous investigation presented in Ref. [32]. We find a regime of “mixed” synchronization with two stable synchronization states, and we explore its classical-to-quantum crossover in Sec. 2.7. In Sec. 2.8, we give an overview of the different synchronization regimes. Finally, in Sec. 2.9, we discuss the effects of thermal noise as compared to quantum noise. This is important to gauge the potential of observing the quantum noise effects discussed here in future experiments.

2.2 Brief review: Classical synchronization of optomechanical oscillators

In this section we briefly review the concepts of classical synchronization of optomechanical systems in the absence of noise. This will set the stage for the discussion of our results on synchronization in the presence of thermal and quantum noise.

A widely studied model for synchronization is the Kuramoto model [11, 14], which describes a set of coupled phase oscillators. Each phase oscillator has a phase $\phi_i$ and an in-
2. (Quantum) Noise-induced phenomena in optomechanical synchronization

Figure 2.1: Synchronization of optomechanical oscillators. Schematics of the setup we study (a) and possible experimental implementations (b)-(d). (b) Micro-disk oscillators that support optical whispering gallery modes which couple evanescently to each other [39, 42], (c) optical racetrack resonator coupled to two nanomechanical oscillators [40], (d) optomechanical crystal structure with two optomechanical cells in the vicinity of each other, allowing for optical and mechanical coupling [157]. (This figure has previously been published in [29].)

Intrinsic frequency $\omega_i$, and couples to the other oscillators via the phase difference. For two oscillators, the two corresponding phase equations collapse into one equation for the relative phase $\delta \phi = \phi_2 - \phi_1$,

$$\delta \dot{\phi} = (\omega_2 - \omega_1) - k \sin(\delta \phi). \quad (2.1)$$

The two oscillators are synchronized if their respective phase velocities become equal, $\dot{\phi}_1 = \dot{\phi}_2$, i.e. $\delta \phi = \text{const}$. From Eq. (2.1) one finds the synchronization threshold: The two oscillators are synchronized if the coupling $k$ exceeds the natural frequency difference, $|k| > |\omega_2 - \omega_1|$. Following this condition, oscillators with identical intrinsic frequencies $\omega_i$ are always synchronized. There is only a single stable value of $\delta \phi$ in the synchronized regime. For $k \to +\infty$, this value approaches zero, $\delta \phi \to 0$, whereas $\delta \phi \to \pi$ for $k \to -\infty$.

Although synchronization appears in a large range of systems which are very different in terms of microscopic parameters, their behaviour can often be captured by effective phase equations of the Kuramoto-type [11, 12]. In the context of optomechanics, the mechanical motion of a single optomechanical system near the Hopf bifurcation can be described with a phase and an amplitude equation [123]. Starting from these equations, an effective Kuramoto-type model for coupled optomechanical systems has been derived [141]. This model describes arrays of arbitrary many optomechanical cells with arbitrary intrinsic frequencies. For two oscillators, the phase equations of this model reduce, again, to a single equation for the phase difference

$$\delta \dot{\phi} = (\omega_2 - \omega_1) - 2S_1 \sin(\delta \phi) - 4S_2 \sin(2\delta \phi). \quad (2.2)$$
Here, $S_1$ and $S_2$ are effective parameters depending on the microscopic parameters of the underlying optomechanical systems [32, 141]. Note that the $S_1$-term was added to the model only later [32], and accounts for the change of the intrinsic frequencies $\omega_i$ with the mechanical oscillation amplitude. Recently, the resulting full Hopf-Kuramoto model has been used to study pattern formation in 2D arrays of optomechanical systems [162]. In contrast to the original Kuramoto model Eq. (2.1), the optomechanical Hopf-Kuramoto model Eq. (2.2) includes a term that involves $\sin(2\delta\phi)$. Rewriting Eq. (2.2) in terms of an effective potential, $\delta\dot{\phi} = -U'(\delta\phi)$ [141], this term corresponds to the appearance of a second minimum close to $\delta\phi = \pi$ which can co-exist with the minimum close to $\delta\phi = 0$. This allows optomechanical systems to synchronize not only in-phase ($0$-synchronization), $\delta\phi \rightarrow 0$, but also anti-phase ($\pi$-synchronization), $\delta\phi \rightarrow \pi$. For the effective potential there are three different possibilities, depending on the parameters $S_1$ and $S_2$: (i) It has a single minimum close to $\delta\phi = 0$ which leads to $0$-synchronization only, (ii) it has a single minimum close to $\delta\phi = \pi$ which leads to $\pi$-synchronization only, or (iii) both minima appear simultaneously in the effective potential, such that the initial conditions determine whether $0$- or $\pi$-synchronization occurs. These three regimes are schematically shown in Fig. 2.2(b) and (c) and discussed in Sec. 2.3.

In general, for optomechanical arrays an effective potential for the phases $\phi_i$ does not exist [162]. However, in the case of two oscillators only, the system can be described with one degree of freedom, i.e. the relative phase $\delta\phi$, and an effective potential for $\delta\phi$ can always be constructed. Below, we make use of the existence of this effective potential to give an intuitive understanding of synchronization even in the presence of noise.

2.3 Model

We now introduce the model that we investigate throughout the rest of this chapter. We study two optomechanical systems which are mechanically coupled, see Fig. 2.1(a). Our goal is to analyze the synchronization behaviour in the presence of quantum and thermal noise. Each optomechanical system consists of a driven optical mode ($\hat{a}_j$) coupled to a mechanical mode ($\hat{b}_j$) via radiation pressure. In a frame rotating at the laser drive frequency $\omega_L$, the Hamiltonian of each optomechanical system is

$$\hat{H}_j = -\hbar \Delta \hat{a}_j^\dagger \hat{a}_j + \hbar \Omega \hat{b}_j^\dagger \hat{b}_j - \hbar g_0 \hat{a}_j^\dagger \hat{a}_j \left( \hat{b}_j + \hat{b}_j^\dagger \right) + \hbar \alpha_L (\hat{a}_j^\dagger + \hat{a}_j).$$

Here, $\Delta = \omega_L - \omega_c$ denotes the detuning from the cavity resonance $\omega_c$, $\Omega$ is the resonance frequency of the mechanical mode, $g_0$ denotes the optomechanical single photon coupling strength, and $\alpha_L$ is the laser driving strength. Note that the optical and the mechanical systems experience damping at a rate $\kappa$ and $\Gamma$, respectively, and these are described by adding to the Hamiltonian a system-bath coupling in the usual manner: $\hat{H}_j^{\text{full}} = \hat{H}_j + \hat{H}_j^{\text{diss}}$.

Driving a single optomechanical system with a blue-detuned laser causes self-oscillations of the mechanical resonator when the optomechanically induced negative damping $\Gamma_{\text{opt}}$ is larger than the intrinsic damping of the oscillator $\Gamma$ [123]. In Fig. 2.2(a) we show this thresh-
Figure 2.2: Threshold of self-sustained oscillations and synchronization in the absence of noise. (a) shows the mechanical energy (related to the limit-cycle amplitude) of an optomechanical system as a function of the laser drive $\alpha_L$ and the detuning $\Delta$. The red line indicates the threshold of self-oscillations, $\Gamma + \Gamma_{\text{opt}} = 0$. The green line separates a region of optical multi-stability, the dashed white line indicates a region where the self-oscillations show strong amplitude modulation. (b) and (c) show schematic pictures of the classical, noiseless synchronization regimes of two coupled optomechanical systems. In (b) we use a small mechanical coupling $K/\Omega_1 = 0.05$ and different mechanical resonance frequencies, $\delta \Omega = 0.075$ to give an overview about all possible synchronization regimes. In (c) we use $K/\Omega_1 = 0.15$ and $\delta \Omega = 0$ instead, which significantly changes the phase diagram. The dashed lines mark regions with even more complex behaviour, not necessarily showing synchronization. [Parameters: mechanical damping $\Gamma/\Omega_1 = 0.015$, optical damping $\kappa/\Omega_1 = 0.3$, optomechanical coupling $g_0/\Omega_1 = 0.3$.] (This figure has previously been published in [29].)

old of self-oscillations. Throughout this work we only consider parameters such that the single optomechanical systems are above this threshold. Self-oscillations at $\Delta < 0$ occur due to the static optomechanical shift which leads to an effective blue detuning, i.e. $\Delta_{\text{eff}} > 0$. These limit-cycle oscillations, in the absence of noise, can effectively be described by a fixed amplitude and a phase and are treated as a prerequisite for synchronization throughout this work. We consider two self-oscillating optomechanical systems that are coupled mechanically with strength $K$, such that the total Hamiltonian of the system (except for the dissipative part) reads

$$\hat{H}_{\text{tot}} = \sum_{j=1,2} \hat{H}_j - \hbar K \left( \hat{b}_1 + \hat{b}^\dagger \right) \left( \hat{b}_2 + \hat{b}_2^\dagger \right).$$

(2.4)

Experimentally, this coupling between the mechanical oscillators can also be mediated by an optical coupling, cf. Fig. 2.1(b)-(d). In recent experiments [39, 40, 42], a single joint optical mode was employed to couple the mechanical oscillators. However, this mode served a dual
2.4 Methods

In experiments, the typical mode of operation is to have the two optomechanical oscillators at slightly different intrinsic mechanical frequencies. These start out un-synchronized but can synchronize upon changing some parameter (e.g., the laser drive strength, the detuning, or potentially the coupling). This is schematically shown in Fig. 2.2(b), where we indicate the synchronization regimes in the absence of noise. In accordance to the Hopf-Kuramoto model, cf. Sec. 2.2, there are unsynchronized regions and three different synchronization regimes: (i) 0-synchronization, (ii) π-synchronization, and (iii) classical bistable synchronization where the type of synchronization depends on the initial conditions. Note that for different intrinsic mechanical frequencies, \( \delta \Omega = \Omega_2 - \Omega_1 \neq 0 \), the relative phase \( \delta \phi \) is not exactly 0 or \( \pi \) but only close to one of these values and varies within the synchronization regime.

However, in the presence of noise it is already interesting to investigate the behaviour even for identical frequencies. In particular, for large-scale optomechanical arrays of identical oscillators, it has been found that there is a synchronization transition as a function of noise strength [32]. Moreover, in the present article we will focus on noise-induced transitions between various synchronization states. The observation of this physics does not rely on whether there is an actual synchronization transition at lower values of the coupling. Therefore, in most of our analysis, we will focus on identical systems, i.e. we assume all the parameters to be equal in both systems. In Fig. 2.2(c) we schematically show the synchronization regimes for identical optomechanical systems and for a larger mechanical coupling \( K \) than in Fig. 2.2(b), but still in the absence of noise. It is important to note that both the mechanical detuning \( \delta \Omega \) and the coupling \( K \) have an influence on this diagram: Not synchronized regions can become synchronized and synchronization regime borders are shifted.

We will comment on the dynamics of two coupled optomechanical oscillators with different frequencies in Sec. 2.8.

2.4 Methods

In this work we use Langevin equations and quantum jump trajectories to study the system described by Hamiltonian (2.4) in the presence of (quantum) noise. Here we want to briefly present both approaches and discuss their respective advantages and problems.

Most of our results are computed with semi-classical Langevin equations. They are obtained by first deriving quantum Langevin equations from Hamiltonian (2.4) using input-output theory [83]. We then adopt a semi-classical approach by turning the quantum Langevin equations into classical Langevin equations for the complex amplitudes \( \alpha_j \) and \( \beta_j \), where the noise terms mimic the quantum-mechanical zero-point fluctuations. This can be understood as a variant of the “truncated Wigner approximation”. The semi-classical equations are:
\[ \dot{\alpha}_1 = \left( i\Delta - \frac{\kappa}{2} \right) \alpha_1 + ig_0 \alpha_1 (\beta_1^* + \beta_1^*) - i\alpha_L - \sqrt{\kappa} \alpha_{1\text{in}}, \]
\[ \dot{\beta}_1 = - \left( i\Omega + \frac{\Gamma}{2} \right) \beta_1 + ig_0 |\alpha_1|^2 + iK\beta_2 - \sqrt{\Gamma} \beta_{1\text{in}}. \]

(2.5)

The corresponding equations for the second optomechanical system can be obtained from Eqs. (2.5) by exchanging the indices $1 \leftrightarrow 2$. Here, $\alpha_{j\text{in}}(t)$ and $\beta_{j\text{in}}(t)$ represent the optical and mechanical input noise, given by Gaussian stochastic processes. Since complex numbers commute, they obviously cannot correctly fulfill the input-output quantum noise correlators, \( \langle \hat{a}_{j\text{in}}^\dagger(t) \hat{a}_{j\text{in}}(t') \rangle = \langle \hat{a}_{j\text{in}}(t) \hat{a}_{j\text{in}}^\dagger(t') \rangle = \delta(t - t') \). Instead, $\alpha_{j\text{in}}$ and $\beta_{j\text{in}}$ are made to mimic quantum noise by fulfilling \( \langle \alpha_{j\text{in}}(t) \alpha_{j\text{in}}^*(t') \rangle = \langle \alpha_{j\text{in}}^*(t) \alpha_{j\text{in}}(t') \rangle = \delta(t - t')/2 \) (and likewise for $\beta_{j\text{in}}$ for $T = 0$). This approach allows to study also large parameter ranges with relatively low computational effort.

Deep in the quantum regime it is initially not clear that these semi-classical Langevin equations describe the correct physical behaviour. In order to verify the qualitative effects observed with Langevin equations, we also present a few results obtained with quantum jump trajectories [178, 179], i.e. an “unraveling” of the Lindblad master equation. Applying this method, the fully quantum system is simulated on an appropriately truncated Hilbert space. Notably, it allows to work with wave functions, in contrast to the formalism of the Lindblad master equation which requires the use of density matrices. Hence, simulations of larger Hilbert spaces become feasible. In addition, quantum jump trajectories give access to additional observables, for instance the full counting statistics. For these reasons, quantum jump trajectories have been applied in many different contexts. In the field of quantum synchronization, such methods have been used to study synchronization of qubits [23]. In the field of cavity optomechanics, they have been employed for instance to discuss QND measurements, photon statistics and single photon optomechanics [110, 115, 180, 181]. Another motivation is the recent experimental detection of individual phonons in optomechanical systems [182] - a step towards monitoring full quantum jump trajectories in experiments.

To explain this approach [178, 179], let us consider for a moment photon decay in cavity 1. The “unraveling” discussed here corresponds to the physical setup of placing a single photon detector at the output port of the cavity. At each time step $\delta t$, the probability of a single photon to leak out of the cavity (at temperature $T = 0$) is given by $p = \kappa \delta t \langle \hat{a}_1^\dagger \hat{a}_1 \rangle$. In the case of a photon loss it is detected at the output port and the wave function is updated \( |\Psi(t + \delta t)\rangle = \hat{a}_1 |\Psi(t)\rangle \). If no photon was lost to the environment, the wave function evolves in time according to a (non-Hermitian) Hamiltonian that is obtained by adding a term $-i\hbar(\kappa/2) \hat{a}_1^\dagger \hat{a}_1$. This additional term accounts for the information gained about the system by not observing a photon [183]. In both cases the state $|\Psi(t + \delta t)\rangle$ has to be normalized before proceeding to the next time step. The treatment of photon loss in cavity 2 and of phonon losses works analogously. This simulation approach naturally accounts for quantum fluctuations.

Although it would be favourable to use quantum jump trajectories throughout the whole
study, this approach is only computationally feasible whenever the needed Hilbert space remains sufficiently small. This leads to severe limitations in the choice of parameters and especially gives no access to the full classical-to-quantum crossover that is studied in Sec. 2.7. In Ref. [127] it has been shown for a single optomechanical system that semi-classical Langevin equations produce good agreement with the full quantum theory. A systematic comparison for two coupled optomechanical systems is not possible, since the number of required Hilbert space dimensions is squared as compared to the single optomechanical system. We comment on this in Sec. 2.7.1 (cf. Fig. 2.4(a) and the discussion below).

Note that quantum jump trajectories serve here as a numerical approach only. In experiments, single photon and phonon detection is not necessary. Instead, the mechanical oscillators could be measured using standard homodyning techniques.

### 2.5 Main results

In this section we briefly state our main results, which will be discussed in the following sections. Most notably, it is known even from the classical theory that two coupled optomechanical oscillators can be in either one of two synchronization states (with a phase difference near 0 or near \( \pi \)). We find a regime of “mixed” synchronization, where transitions between 0- and \( \pi \)-synchronization occur (Sec. 2.6). These transitions are driven by (quantum or thermal) noise and cannot be found in the classical, noiseless situation. The average residence times in the two synchronization states can differ and their ratio varies with the system parameters (Sec. 2.7). Investigating the classical-to-quantum transition, we find that mixed synchronization can evolve from two different regimes in the classical, noiseless limit: (i) there are already two stable synchronization states but in the absence of noise there are no transitions, (ii) there is only one stable synchronization state and only the presence of noise leads to a second stable solution. Although the first sections are devoted to the investigation of quantum noise effects, we note that we find similar effects for thermal noise acting on the mechanical resonator. However, quantitative differences remain due to the different nature of the noise source (Sec. 2.9). We find that quantum noise effects should dominate over thermal noise effects if the optomechanical cooperativity is sufficiently large, and a large value of \( g_0 \) is not necessarily required.

### 2.6 Multistable quantum synchronization

First, we start by analyzing two coupled optomechanical systems, cf. Fig. 2.1(a), deep in the quantum regime. Quantum jump trajectories are used to initially investigate the full quantum dynamics. In the following, we consider a small-amplitude limit cycle and a large single-photon coupling strength, \( g_0/\kappa = 1 \). This ensures that quantum fluctuations can potentially have a large impact on the system’s dynamics. Furthermore, small photon and phonon num-
Figure 2.3: Multistability in optomechanical quantum synchronization. Distribution of the relative phase $\delta \phi$ in different synchronization regimes, a typical sample of a corresponding quantum jump trajectory starting in the steady state, and a sketch of the corresponding effective potential. (a) shows 0-synchronization, (b) shows $\pi$-synchronization, and (c) shows mixed synchronization. A rotating wave approximation for the mechanical coupling has been used. [Parameters: (a) mechanical coupling $K/\Omega = 0.3$, mechanical damping $\Gamma/\Omega = 0.015$; (b) $K/\Omega = 0.15$, $\Gamma/\Omega = 0.01$; (c) $K/\Omega = 0.15$, $\Gamma/\Omega = 0.015$; other parameters are: optical damping $\kappa/\Omega = 0.3$, laser driving strength $\alpha_L/\Omega = 0.3$, optomechanical coupling $g_0/\kappa = 1$, and optical detuning $\Delta/\Omega = 0.15$.] (This figure has previously been published in [29].)
tration of the uncoupled self-oscillators in the absence of noise can effectively be described by \( \langle \hat{b}_j \rangle(t) \approx |b|e^{-i(\Omega t + \phi_0)} \), with an initial random phase \( \phi_0 \). Thus, the correlator \( \langle \hat{b}_1^\dagger \hat{b}_2 \rangle \sim e^{-i\delta \phi} \) can be interpreted as a measure for the relative phase \( \delta \phi \). In Fig. 2.3 we show the probability density of the relative phase obtained from the distribution of \( \delta \phi \) and a typical sample of the corresponding phase trajectory. We find a 0-synchronized regime, Fig. 2.3(a), where the relative phase is predominantly close to \( \delta \phi = 0 \). Similarly, we find a \( \pi \)-synchronized regime, Fig. 2.3(b), for different parameters. As expected, noise prevents perfect synchronization and thus the corresponding maximum in the probability density has a finite width. Likewise, the trajectories show fluctuations around either \( \delta \phi = 0 \) or \( \delta \phi = \pi \). These two synchronization regimes have an analog in the classical, noiseless limit, cf. Sec. 2.2. In addition, we find another regime where the probability density of the phase difference has maxima close to both \( \delta \phi = 0 \) and \( \delta \phi = \pi \), see Fig. 2.3(c). The corresponding trajectory shows that in this regime transitions between the two synchronization states occur. We call this regime “mixed” synchronization. This is in contrast to the classical, noiseless result, where the system does not change its synchronization state with time, not even in the regime of bistable synchronization.

Similar to the classical, noiseless system [141] we can understand these results in terms of an effective potential for the relative phase \( \delta \phi \). This is illustrated in the bottom row of Fig. 2.3. It offers an intuitive understanding of our results. The noise makes the phase fluctuate around the stable point(s) near \( \delta \phi = 0 \) (and)or \( \delta \phi = \pi \). In the case of mixed synchronization the effective potential has two minima, near \( \delta \phi = 0 \) and \( \delta \phi = \pi \), and quantum noise drives transitions between those two states. The probability to be in either the 0- or \( \pi \)-synchronized state is given by the area of the corresponding maximum in the probability density. It is associated to the depth of the minimum in the effective phase potential. The ratio between the probabilities of the two states can assume arbitrary values, depending on the parameters of the system. Note that the absence of two distinct peaks in the probability density does not necessarily mean that the system explores the region around one synchronization state only. In fact, sometimes the trajectories themselves can reveal short stretches of phase dynamics in the vicinity of the other state. Nevertheless, if the fraction of time spent in that other state is short, a second peak will not be visible.

From this interpretation in terms of an effective potential, it is clear that the classical bistable synchronization regime where two potential minima already exist (but no transitions can occur), turns into a mixed synchronization regime (showing transitions) in the presence of noise. However, our analysis in the following section reveals that mixed synchronization can also appear for parameters where classically there is only a single stable synchronization state.

### 2.7 Classical-to-quantum crossover

At the moment, the single-photon coupling strength \( g_0 \) is still comparatively small in almost all experiments. As a consequence, quantum effects have only been observed in the
linearized regime where only Gaussian states are produced. In the most promising cases [86, 184], \( g_0/\kappa \) can take values up to \( 10^{-2} \) and \( g_0/\Omega \) up to above \( 10^{-4} \). Much larger values have been reported for experiments with cold atoms (up to around \( g_0/\kappa \sim 1 \)) [98], but these do not operate in the “good cavity limit”, i.e. one has \( \kappa \gg \Omega \) in those experiments, precluding the observation of single-photon strong coupling effects. Nevertheless, as experiments are approaching the single-photon strong coupling regime, they will gradually see increasingly strong effects of quantum fluctuations even in nonlinear dynamics. In this section, it is our aim to explore the crossover between the classical regime (small \( g_0/\kappa \)) and the quantum regime (large \( g_0/\kappa \)) with respect to optomechanical quantum synchronization. We will focus our investigations mostly on the mixed synchronization regime which is the most interesting one, as we can have noise-induced transitions between the synchronization states.

In this section, we will disregard thermal noise, i.e. we assume temperature \( T = 0 \), such that only quantum noise is present. The “classical” regime we are discussing here is therefore the noiseless limit of the classical equations of motion. We will later remark on the effects of thermal noise (Sec. 2.9).

To explore the classical-to-quantum crossover, we want to effectively vary \( \hbar \) while making sure to keep all the classical predictions unchanged. Notably, the optomechanical coupling strength \( g_0 = \partial \omega_c / \partial x \) depends on \( \hbar \). For the simplest case of a Fabry-Pérot cavity of length \( L \) with one static and one movable mirror the optomechanical coupling is \( g = \omega_c(0)x_{ZPF}/L \sim \sqrt{\hbar} \), where \( x_{ZPF} = \sqrt{\hbar/2m\Omega} \) denotes the zero-point fluctuations of the mechanical oscillator of mass \( m \). As discussed in Ref. [127], the “quantum parameter” \( g_0/\kappa \sim \sqrt{\hbar} \) can thus be varied to effectively change the quantum noise strength. This implies that all classical (\( h \)-independent) parameters \( (\kappa/\Omega, \Gamma/\Omega, \Delta/\Omega, K/\Omega, g_0\alpha_L/\Omega^2) \) are kept fixed while \( g_0 \) is modified. To see that the quantum parameter has indeed this anticipated effect, it is very helpful to rescale the amplitudes \( \alpha_j \) and \( \beta_j \), Eqs. (2.5), such that they tend to a well-defined finite value in the classical limit \( g_0/\kappa \rightarrow 0 \) [127]. This can be achieved by defining \( \tilde{\alpha}_j = g_0\alpha_j \) and \( \tilde{\beta}_j = g_0\beta_j \). While \( |\alpha_j|^2 \) gives the number of photons (a “quantum-mechanical” quantity), the rescaled version \( |\tilde{\alpha}_1|^2 = g_0^2 |\alpha_1|^2 \sim h|\alpha_1|^2 \) is proportional to the energy \( h\omega_c |\alpha_1|^2 \) inside the cavity (i.e. a classical quantity). A similar argument applies to \( \tilde{\beta}_j \). With this rescaling, we find the following equations:

\[
\begin{align*}
\dot{\tilde{\alpha}}_1 &= \left( i\Delta - \frac{\kappa}{2} \right) \tilde{\alpha}_1 + i\tilde{\alpha}_1 (\tilde{\beta}_1 + \tilde{\beta}_1^*) - i\tilde{\alpha}_L - \sqrt{\hbar} \tilde{\alpha}_{1\text{lin}}, \\
\dot{\tilde{\beta}}_1 &= -\left( i\Omega + \frac{\Gamma}{2} \right) \tilde{\beta}_1 + i|\tilde{\alpha}_1|^2 + iK\tilde{\beta}_2 - \sqrt{\hbar} \tilde{\beta}_{1\text{lin}}.
\end{align*}
\]

Here \( \tilde{\alpha}_L = g_0\alpha_L \) is the rescaled laser-driving amplitude that we keep fixed while varying \( g_0 \). The important observation here is that \( g_0 \) has been completely eliminated from the equations and now only appears in the strength of the quantum noise: we now have \( \langle \tilde{\alpha}_{jin}(t)\tilde{\alpha}_{jin}^*(t') \rangle = g_0^2 \delta(t-t')/2 \) (and likewise for \( \tilde{\beta}_{jin} \)) which indeed vanishes in the classical limit of \( g_0 \sim \sqrt{\hbar} \rightarrow 0 \).

If we consider mechanical thermal noise at finite temperature \( T \) (as discussed in Sec. 2.9), we have \( \langle \tilde{\beta}_{jin}(t)\tilde{\beta}_{jin}^*(t') \rangle = g_0^2(n_{\text{th}} + 1/2)\delta(t-t') \). Here, \( n_{\text{th}} \) denotes the thermal occupancy of...
2.7. Classical-to-quantum crossover

Figure 2.4: Quantum-to-classical crossover. (a) Synchronization measure as a function of the quantum parameter $g_0/\kappa$, using Langevin equations (black line), classical (noiseless) Langevin equations (black circle, $g_0/\kappa = 0$), and with quantum jumps (red triangles). (b) and (c) show trajectories of the relative phase in the quantum regime $g_0/\kappa = 1$ (blue square) and for $g_0/\kappa = 0.4$ (green dot). Other parameters are as in Fig. 2.3(c), a rotating wave approximation has been used for the mechanical coupling. (This figure has previously been published in [29].)

the bath coupled to the mechanical oscillator. The product $g_0^2 n_{th} \sim \hbar n_{th}$ becomes independent of $\hbar$ in the classical limit $k_B T \gg \Omega$, where $n_{th} \approx k_B T/\Omega$ and $k_B$ is Boltzmann’s constant. In summary, the rescaled equations (2.6) nicely show how an increase of $g_0$ can indeed be viewed solely as an increase of the strength of “quantum noise” in our system.

Note that if $g_0 \to 0$ we have to increase the laser driving strength $\alpha_L$ to keep $\tilde{\alpha}_L = g_0 \alpha_L$ constant which means that the total light energy circulating inside the optical cavity is constant. Due to $\hbar \to 0$, this corresponds to an increasing number of photons inside the cavity and thus increases drastically the size of the Hilbert space necessary for the full quantum simulation. Therefore, quantum jump trajectories are not suitable to explore the full quantum-to-classical crossover and we have to apply Langevin equations.

2.7.1 Synchronization as a function of quantum noise strength

In all studies of synchronization, one needs to select suitable quantities that measure the degree of synchronization. In this context, it is important to note that for any finite noisy system (subject to quantum and/or thermal noise), there is no sharp synchronization transition
and correspondingly no unambiguous measure that displays nonanalytic behaviour at any parameter value. Before proceeding to our results, we summarize and discuss the synchronization measures adopted here which have to be combined to obtain a full picture: (i) the probability density of $\delta \phi$, (ii) the average phase factor $\langle e^{-i\delta \phi} \rangle$, and (iii) individual trajectories.

The probability density of the relative phase $\delta \phi$ is either mostly flat (no synchronization) or, as shown in Fig. 2.3, centered predominantly around 0 or $\pi$, or it may have two peaks, according to the synchronization regime. In the following, we aim to compress the information contained in the phase distribution into one quantity and calculate the normalized correlator $C = \langle \hat{b}_1^\dagger \hat{b}_2 \rangle / \sqrt{\langle \hat{b}_1^\dagger \hat{b}_1 \rangle \langle \hat{b}_2^\dagger \hat{b}_2 \rangle} \approx \langle e^{-i\delta \phi} \rangle$. Its real value, $\text{Re}[C] = \langle \cos \delta \phi \rangle$, distinguishes the three different synchronization regimes: (i) $\langle \cos \delta \phi \rangle \approx 1$ for 0-synchronization, (ii) $\langle \cos \delta \phi \rangle \approx -1$ for $\pi$-synchronization, (iii) intermediate values of $\langle \cos \delta \phi \rangle$ for mixed synchronization. However, this measure has its limitations: When $\delta \phi$ is more or less evenly distributed (no synchronization) $\langle \cos \delta \phi \rangle \approx 0$, this cannot be distinguished from a mixed synchronization situation where almost equal time is spent in the 0- and $\pi$-synchronized states. Furthermore, even in the absence of synchronization (and even in the noiseless case) the phase $\delta \phi$ may spend an increased amount of time around certain values. This leads to a finite value of $\langle \cos \delta \phi \rangle$, and similarly would also show up in the phase distribution. A solution to this problem is to simultaneously look at a part of the corresponding trajectory, where synchronization can easily be distinguished from an unsynchronized state. Instead, one could start to use more complicated correlators, e.g. $\langle \hat{b}_1^\dagger \hat{b}_1 \hat{b}_2 \hat{b}_2 \rangle \sim e^{-i2\delta \phi}$. This correlator allows to distinguish unsynchronized states from synchronized states, but an additional measure is needed to distinguish 0- from $\pi$-synchronization. Note that the imaginary part of the above defined correlator, $\text{Im}[C] = \langle \sin \delta \phi \rangle$, can be used as well. However, in the special case of identical optomechanical systems $\text{Im}[C] \approx 0$ due to the symmetry of the system.

In Fig. 2.4(a) we show how $\langle \cos \delta \phi \rangle$ varies as a function of the quantum parameter $g_0/\kappa$. We chose parameters that lead to a mixed synchronization regime for larger values of $g_0/\kappa$. In the deep quantum regime ($g_0/\kappa \rightarrow 1$) the probability $P_0$ to find the system in the 0-synchronized state is larger than the probability $P_{\pi}$ to find the phase around $\pi$, such that $\langle \cos \delta \phi \rangle > 0$. Going towards smaller values of $g_0/\kappa$, the ratio $P_{\pi}/P_0$ increases, such that eventually $\langle \cos \delta \phi \rangle < 0$. Finally, we should reach the classical (noiseless) limit, when $g_0/\kappa \rightarrow 0$. It turns out that, for the parameters adopted here, the classical solution always ends up in the $\pi$-synchronized state, independent of initial conditions. This implies that there is only one minimum in the effective potential. We conclude that the system has turned from a mixed synchronization regime into a purely $\pi$-synchronized regime as the quantum parameter was reduced. This cannot be understood in the simple picture of a noise-independent phase potential. We will discuss this kind of behaviour in more detail later on (Sec. 2.7.3).

Note that the calculations for Fig. 2.4 have been performed using Langevin equations; although in the deep quantum regime, two data points were also acquired with quantum jump trajectories (red triangles). They are shifted as compared to the Langevin results, but show the same trend. We expect that the difference between the Langevin and quantum jump results decreases for smaller quantum parameters $g_0/\kappa$, as it is the case for a single
optomechanical system [127]. Since smaller $g_0/\kappa$ require a significantly larger Hilbert space for the quantum jump simulations, we cannot compute this for our coupled system. For large quantum parameter $g_0/\kappa \sim 1$, qualitative differences between the full quantum model and the Langevin equations have been already observed in Ref. [127] as well. Especially a shift of the detuning $\Delta$ was reported, that could be determined numerically also for our system. Taking this detuning shift into account would improve the agreement of our results, although differences remain. Here, we show the uncorrected outcomes of both approaches.

2.7.2 Residence times in the mixed synchronization regime

The measure $\langle \cos \delta \phi \rangle$ quantifies the fraction of total time spent in 0-synchronized parts as compared to $\pi$-synchronized parts. However, it does not provide any information about the rate of transitions between the two synchronization states. Based on the effective potential
picture, one might expect the transition rates to be determined by the barrier height and the noise strength. In particular, for larger effective noise strengths $g_0/\kappa$, we expect more frequent transitions. This behaviour is qualitatively visible in Fig. 2.4(b) and (c). However, as concluded in the previous section, the potential picture is not sufficient to explain all observations. Thus, we now turn to a quantitative analysis and discuss how the transition rates between 0- and $\pi$-synchronized states (i.e. the typical residence times $\bar{\tau}$) change during the classical-to-quantum crossover. We extract the fluctuating residence times from the phase trajectories and obtain their distribution. The results are shown in Fig. 2.5(a) and (b) for the 0- and $\pi$-synchronized states, for two different quantum parameters $g_0/\kappa$. In all cases the probability densities decay exponentially with time, $\sim e^{-\tau/\bar{\tau}}$, and the average residence time $\bar{\tau}$ is obtained from a fit to the distribution. The extracted average residence times $\bar{\tau}_0$ and $\bar{\tau}_\pi$ for the two states are shown in Fig. 2.5(c) as a function of the quantum parameter. Note that the ratio of residence times equals the ratio of probabilities, $\tau_0/\tau_\pi = P_0/P_\pi$. Nevertheless, the dependence of the times $\tau_0, \tau_\pi$ on $g_0/\kappa$ reveals new information.

We have chosen parameters such that at $g_0/\kappa = 1$ both 0-synchronized and $\pi$-synchronized parts have almost equal average residence times. This corresponds to $\langle \cos \delta \phi \rangle \approx 0$ and $P_0 \approx P_\pi$. Furthermore, in the classical limit $g_0/\kappa = 0$ the system is $\pi$-synchronized only. When the classical limit $g_0/\kappa \rightarrow 0$ is approached, we find that both $\tau_0$ and $\tau_\pi$ increase. As expected, $\tau_\pi$ increases much faster than $\tau_0$ and eventually diverges for $g_0/\kappa \rightarrow 0$, as the system gets trapped forever in the $\pi$-state. In contrast, $\tau_0$ increases first when decreasing $g_0/\kappa$, but then saturates at a finite level. Such a behaviour is unexpected based on the simple phase potential picture, where a fixed potential would imply diverging residence times for both states in the noiseless limit. The behaviour observed here hinges on the fact that the synchronization regime switches from “mixed” to “$\pi$” as one reduces the quantum parameter (i.e. reduces the quantum noise). The observations would change significantly for different parameters, where the system always stays in the mixed regime, for any value $g_0/\kappa > 0$. Then, one expects the simple picture of a fixed phase potential to be approximately correct and both residence times to diverge as the noise is becoming weaker.

### 2.7.3 Noise-induced synchronization bistability

In the previous sections we have explained that some basic features of the classical-to-quantum crossover, like the increase of the residence times with decreasing quantum parameter, can be understood as effects of a decreasing quantum noise strength. The decrease of noise strength leads to less frequent transitions across an energy barrier in the effective potential. However, we made also less easily explained observations: (i) the reverse in the order of $\tau_0$ and $\tau_\pi$ as $g_0$ is being reduced, (ii) the saturation of $\tau_0$ at low noise levels, (iii) the disappearance of stable 0-synchronization (for the applied parameters) in the classical limit $g_0/\kappa = 0$. Whereas (i) and (ii) could originate from more complicated potential shapes with a combination of broad and narrow minima, (iii) suggests that the effective potential itself changes when the quantum parameter is varied. In Fig. 2.6 we show how the distribution of
2.8 Overview of synchronization regimes

In the previous sections we have shown examples of the different synchronization regimes in the presence of quantum noise and studied the properties of mixed synchronization in more detail. In the following, we map out the different synchronization regimes as a function of

\[ \frac{g_0}{\kappa} \]

\[ \delta \phi \]

Figure 2.6: Appearance of bistability. The distribution of the relative phase as a function of the quantum parameter \( g_0/\kappa \). Parameters are as in Fig. 2.5, but with mechanical coupling \( K/\Omega = 0.1 \). (This figure has previously been published in [29].)

\( \delta \phi \) evolves as a function of \( g_0/\kappa \). For very small values of \( g_0/\kappa \), there is only a single peak close to \( \delta \phi \approx \pi \), in accordance with the single stable solution of the classical limit. While increasing the quantum parameter, this peak is first broadened. The increasing quantum noise strength allows the system to explore more of the effective potential around the minimum. A significant accumulation close to \( \delta \phi \approx 0 \) appears only for rather large quantum parameters, signaling the appearance of a second stable solution, i.e. a second minimum in the effective potential at \( \delta \phi = 0 \).

In addition to the above described appearance of a second stable solution, there are also parameter regions where already in the classical regime the effective potential has minima at both \( \delta \phi = 0 \) and \( \delta \phi = \pi \) (classical bistable synchronization). In this case quantum noise naturally drives transitions between the two synchronization states as soon as it is added to the description of the system. The number of observed transitions then naturally depends on both the noise strength as well as the potential shape.
the mechanical coupling $K$ and the quantum parameter $g_0/\kappa$. Furthermore, we also discuss the case of detuned mechanical oscillators.

Figure 2.7(a) shows the synchronization measure $\langle \cos \delta \phi \rangle$ for resonant oscillators. We have indicated the synchronization regimes (which are not sharply delineated). For the applied parameters, we find 0-synchronization for large mechanical coupling $K$ in both the quantum and classical regime. In the classical, noiseless limits $\langle \cos \delta \phi \rangle \rightarrow 1$, indicating less fluctuations around the synchronization state. In contrast, at smaller mechanical coupling, we find more complicated behaviour: there is mixed synchronization for $g_0/\kappa \sim 1$, while the classical limit $g_0/\kappa \rightarrow 0$ selects either 0- or $\pi$-synchronization, depending on the mechanical coupling $K$. The “pixelated” region in Fig. 2.7(a) indicates that the system is multistable even in the classical limit. There, the residence times have become so large that the system is stuck in a random synchronization state depending on initial conditions and the transient behaviour. Notably, the closer the system is to a border of synchronization regimes in the classical limit (this can be seen in Fig. 2.7(a) for small $g_0/\kappa$ when $K$ is varied), the smaller the noise strength $(g_0/\kappa)$ that is needed to lead to mixed synchronization. An exception is of course the classically bistable regime, where mixed synchronization appears naturally as soon as there is noise. An interesting feature appears close to $K/\Omega \approx 0.18$, where the measure $\langle \cos \delta \phi \rangle$ shows a sharp dip in the middle of a 0-synchronized region. We suspect a nonlinear resonance, since the oscillator trajectory $x_j(t)$ is no longer simply sinusoidal and period-doubling is observed. At the same time, the oscillation amplitude increases. For larger values of the mechanical coupling $K$, the trajectories are simply sinusoidal again, with the same frequency as for coupling strengths below the feature.

Up to now, we have studied the ideal case of identical mechanical oscillators. We now turn to the case where the mechanical oscillators have slightly different resonance frequencies, i.e. $\delta \Omega = \Omega_2 - \Omega_1 \neq 0$. This is a typical situation in experiments, since fabrication inaccuracies lead to deviations between two systems. Figure 2.7(b) shows how $\delta \Omega$ affects synchronization. A finite $\delta \Omega$ corresponds to a tilted effective potential in the classical, noiseless limit, where a finite threshold for synchronization appears [141]. This classical threshold is indicated in Fig. 2.7(b) with a dashed line. Similar behaviour is visible for quantum parameters $g_0/\kappa > 0$. However, for large $g_0/\kappa$ it is not possible to determine the onset of synchronization, because the measure $\langle \cos \delta \phi \rangle$ cannot distinguish between no synchronization and mixed synchronization. Instead, we now also have to analyze the trajectories in more detail, which reveal mixed synchronization for sufficiently large mechanical coupling. A significant deviation from the classical threshold cannot be observed at the given resolution. We expect the threshold to slightly increase for larger quantum parameter $g_0/\kappa$. At the same time, however, the threshold is also smeared out due to quantum noise.

Here, we have chosen to show the dependence of the synchronization regimes on the mechanical coupling strength $K$ and the quantum parameter $g_0/\kappa$. Note that other parameters
influence the synchronization type as well. In Fig. 2.2 we have already seen that the laser driving strength $a_L$ and the detuning $\Delta$ influence the classical synchronization regimes. Also the mechanical damping $\Gamma$ can affect the observed synchronization, cf. Fig. 2.3(b) and (c). However, first of all it already influences the limit cycles of individual optomechanical systems by modifying the threshold to self-sustained oscillations. In addition, $\Gamma$ has an influence on the mechanical noise strength. Note that, when changing these parameters, care has to be taken to remain on a stable limit cycle for each optomechanical system.

### 2.9 Thermal noise

So far, we assumed zero temperature environments for both the optical and the mechanical mode. In this section we investigate the effects of thermal mechanical noise on synchronization.

For our study we use the Langevin equations (2.5) with modified mechanical noise correlators to account for the coupling of the mechanical oscillators to a finite temperature bath, $\langle \beta_{jn}(t)\beta_{jn}^\dagger(t') \rangle = \langle \beta_{jn}^\dagger(t)\beta_{jn}(t') \rangle = (n_{th} + 1/2)\delta(t - t')$, where $n_{th}$ denotes the thermal occupancy of the bath. Hence, both quantum and thermal noise are included. For optical frequencies in the visible spectrum the effective thermal occupation of the optical bath is very
2. (Quantum) Noise-induced phenomena in optomechanical synchronization

small. Thus, the assumption of an optical bath at zero temperature is valid and the optical input noise terms are not modified.

In Fig. 2.7(c) we show an overview of the synchronization regimes as a function of thermal noise \( n_{th} \). Here, we chose a comparatively small quantum parameter \( g_0/\kappa = 0.01 \) in order to observe mainly effects due to thermal noise. For small \( n_{th} \) the results are similar to Fig. 2.7(a) at small \( g_0/\kappa \). In both cases the influence of the quantum or thermal noise is still weak. For increasing thermal noise strength \( n_{th} \), we find qualitatively the same behaviour as for increasing quantum noise. However, quantitative differences appear with increasing \( n_{th} \): Even though a mixed synchronization regime can develop for both quantum and thermal noise, the evolution of the relative weight of both synchronization states is different.

In the following, we want to estimate the critical thermal noise strength \( n^*_{th} \) at which thermal and quantum noise should have a comparable effect. At lower temperatures, quantum noise will dominate. The main source of quantum fluctuations is the laser shot noise (for the parameters explored here). Thus, we estimate the effect of optical quantum fluctuations on the mechanical oscillator, with the (symmetrized) shot-noise spectrum evaluated at the mechanical resonance frequency \( \Omega \) \([80, 84]\),

\[
S_{FF}^{SN}(\Omega) = \frac{1}{2} \left( \frac{\hbar g_0}{x_{ZPF}} \right)^2 \left\langle \hat{a}^\dagger \hat{a} \right\rangle \left( \frac{\kappa}{(\kappa/2)^2 + (\Omega + \Delta)^2} + \frac{\kappa}{(\kappa/2)^2 + (\Omega - \Delta)^2} \right). \tag{2.7}
\]

We expect similar effects from quantum and thermal noise if the shot-noise spectrum at the mechanical resonance frequency \( \Omega \) becomes equal to the thermal force spectrum, \( S_{FF}^{SN}(\Omega) = S_{FF}^{th} \). The thermal noise spectrum at temperatures \( k_B T \gg \hbar \Omega \) is

\[
S_{FF}^{th} = 2m\Gamma k_B T, \tag{2.8}
\]

where \( T \) is the temperature of the thermal mechanical bath. Setting \( S_{FF}^{SN} \) and \( S_{FF}^{th} \) equal, we find (in the resolved sideband regime \( \kappa \ll \Omega \) at \( \Delta \approx \Omega \)):

\[
n^*_{th} = k_B T/h \Omega = \mathcal{C}/2, \tag{2.9}
\]

where we used the optomechanical cooperativity \([80]\)

\[
\mathcal{C} = \frac{4g_0^2}{x\Gamma \left\langle \hat{a}^\dagger \hat{a} \right\rangle}. \tag{2.10}
\]

This approach suggests that observing quantum noise phenomena does not necessarily require a large \( g_0/\kappa \) and very low temperatures. Instead, if the cooperativity is sufficiently large (comparable to values that enable ground-state cooling, \( \mathcal{C} > n_{th} \)), quantum noise should dominate the behaviour of the system even in the presence of thermal noise.

However, depending on parameters, we find large deviations from this simple expectation. In these cases, the real shot noise spectrum is no longer well described by the weak-coupling expression of \( S_{FF}^{SN}(\Omega) \) given above, and the actual noise strength may have a much larger value. Consequently, the transition between behaviour dominated by quantum noise
vs. that dominated by thermal noise takes place at much larger values of $n_{th}^*$ than those predicted by Eq. (2.9). In other words, it should be even easier to observe quantum noise in optomechanical synchronization than the naive ansatz would lead one to expect.

Here, we don’t observe that quantum noise can be exactly mapped to thermal noise. This is already evident in the rescaled Langevin equations (2.6). Physically, the thermal force spectrum acting on the mechanical oscillator is flat in frequency, whereas the optical shot-noise spectrum is frequency dependent and is also modified by the dynamics of the system.

2.10 Conclusions

We have investigated the effects of quantum and thermal noise on two coupled optomechanical limit-cycle oscillators. One usually expects that noise prevents strict synchronization, i.e. exact phase locking and a sharp transition to synchronization. Here we have shown that fluctuations additionally drive transitions between 0- and $\pi$-synchronization, i.e. the two synchronization states that can appear in the absence of noise. We have discussed the residence times of these states and observed a smooth crossover between different synchronization regimes. Finally, we have compared the effects of quantum and thermal noise. We have argued that it should be possible to experimentally reach the regime where quantum noise dominates. This should happen when the optomechanical cooperativity is large enough for ground state cooling.

For further investigations it would be useful to identify a measure that can genuinely distinguish between an unsynchronized regime and 0-, $\pi$- and mixed synchronization. Finally, it will be very interesting to extend the insights obtained here to large optomechanical arrays.
2. (Quantum) Noise-induced phenomena in optomechanical synchronization
Quantum-coherent phase oscillations in synchronization

In this chapter we present our results that were published in this form in

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and its Supplemental material. Some minor changes have been made to the text in order to better embed this work into this thesis. Furthermore, a former footnote is now added directly to the corresponding caption of Fig. 3.2, and for graphical reasons, we split one figure from the publication into two separate figures, now Figs. 3.3 and 3.4.

Recently, several studies have investigated synchronization in quantum-mechanical limit-cycle oscillators. However, the quantum nature of these systems remained partially hidden, since the dynamics of the oscillator’s phase was overdamped and therefore incoherent. We show that there exist regimes of underdamped and even quantum-coherent phase motion, opening up new possibilities to study quantum synchronization dynamics. To this end, we investigate the Van der Pol oscillator (a paradigm for a self-oscillating system) synchronized to an external drive. We derive an effective quantum model which fully describes the regime of underdamped phase motion and additionally allows us to identify the quality of quantum coherence. Finally, we identify quantum limit cycles of the phase itself.
3. Quantum-coherent phase oscillations in synchronization

3.1 Introduction

Synchronization is commonly studied in so-called limit cycle (LC) oscillators that arise from an interplay of linear and nonlinear effects [12]. For instance, linear amplification causes an instability, whereas nonlinear damping limits the oscillator’s dynamics to a finite amplitude. Notably, the phase remains free, which allows synchronization of the oscillator to an external periodic drive or other LC oscillators. A transition from the intrinsic LC motion towards synchronized oscillations occurs depending on the coupling strength to (and frequency mismatch of) the external reference.

Quantum synchronization, i.e., the study of quantum systems whose classical counterparts synchronize, has recently attracted increasing theoretical attention. So far, studies of quantum synchronization have only explored overdamped phase motion. This implies that the dynamics, although taking place in quantum systems, remains always incoherent and classical-like, ruling out the observation of interesting effects like quantum tunnelling or superposition states of different synchronization phases. In the present article, we discover quantum-coherent phase dynamics.

Theoretical studies of quantum synchronization have been performed for different platforms, including optomechanics [29, 32], atoms and ions [26, 28], Van der Pol (VdP) oscillators [15, 16, 20, 25, 33], and superconducting devices [50, 51]. Measures of synchronization in the presence of quantum noise have been proposed in Refs. [28, 43, 45].

On the experimental side, only classical synchronization has been studied so far for a wide range of systems [11], including more recently optomechanical systems [38–40, 42]. In the well-developed field of classical synchronization, overdamped phase motion is the standard ingredient both of phenomenological equations and microscopically derived models. For example, locking to an external force is described by the so-called Adler equation, a first-order differential equation for the phase. Similarly, synchronized optomechanical systems are described by the first-order phase equation of the Hopf-Kuramoto model [32, 141, 162]. However, it has been noticed that classical synchronization also allows for underdamped phase dynamics. For instance, the classical VdP oscillator features underdamped phase motion and even (synchronized) phase self-oscillations [12, 13, 185, 186]. Both regimes have recently been observed experimentally using a nanoelectromechanical system [187]. Classical phase self-oscillations, also called phase trapping, have also been observed with coupled laser modes [188]. Furthermore, synchronized Josephson junction arrays can be mapped to the Kuramoto model including inertia [11, 189].

Here we will show that a regime of quantum-coherent dynamics exists and that underdamped phase dynamics is a necessary but not sufficient condition to observe this regime. Rather, it is the dynamically generated non-equilibrium dephasing rate that has to become smaller than the oscillation frequency. Additionally, we identify phase self-oscillations in the quantum regime.

We obtain these insights for a paradigmatic model, the quantum version of the VdP oscillator subject to an external drive. The synchronization of the VdP oscillator to this external drive is...
3.2 Quantum model

The quantum VdP oscillator subject to an external drive is described by the master equation \((\hbar = 1)\)

\[
\dot{\hat{\rho}} = -i \left[ -\Delta \hat{b}^\dagger \hat{b} + i F (\hat{b} - \hat{b}^\dagger), \hat{\rho} \right] + \gamma_1 \mathcal{D}(\hat{b}^\dagger) \hat{\rho} + \gamma_2 \mathcal{D}(\hat{b}^2) \hat{\rho},
\]

with \(\mathcal{D}(\hat{O}) \hat{\rho} = \hat{O} \hat{\rho} \hat{O}^\dagger - \{\hat{O}^\dagger \hat{O}, \hat{\rho}\}/2\). Here, \(\Delta = \omega_d - \omega_0\) is the detuning of the oscillator’s natural frequency \(\omega_0\) from the frequency of the external drive \(\omega_d\) and \(F\) is the driving force. The two

![Diagram of quantum synchronization]

Figure 3.1: Quantum synchronization. Steady-state Wigner density \(W_{ss}(x, p)\) and phase probability distribution \(P(\phi)\) of (a) an undriven \((F/\gamma_1 = 0)\) and (b) an externally driven \((F/\gamma_1 = 10)\) VdP oscillator. (a) The ring-shaped Wigner function indicates LC motion. (b) With increasing detuning \(\Delta/\gamma_1\), the synchronization phase changes and synchronization becomes weaker. Parameters: \(\gamma_2/\gamma_1 = 5 \times 10^{-3}\), (a) \(\Delta/\gamma_1 = 0\), (b) “1, 2, 3” correspond to \(\Delta/\gamma_1 = 0, 0.5,\) and 1. (This figure has previously been published in [21].)
dissipative terms in Eq. (3.1) describe gain and loss of one and two quanta at rates $\gamma_1$ and $\gamma_2$, respectively.

In Fig. 3.1 we show the steady-state Wigner function along with the corresponding phase probability distribution $P(\phi) = \sum_{n,m=0}^{\infty} e^{i(m-n)\phi} \langle n|\hat{\rho}_{ss}|m \rangle$ [28] by numerically solving Eq. (3.1) for its steady state $\hat{\rho}_{ss}$. In the absence of an external force ($F=0$), the two competing dissipation rates $\gamma_1$ and $\gamma_2$ lead to LC motion of the VdP oscillator, Fig. 3.1(a). For a finite applied force ($F \neq 0$) and sufficiently small detuning $\Delta$, the VdP oscillator synchronizes to the external force and a fixed phase relation between the VdP oscillator and the force is present. In the rotating frame, this corresponds to a localized Wigner density and a phase distribution $P(\phi)$ with a distinct peak. With increasing detuning, the VdP oscillator is less synchronized to the external force [related to the height and width of $P(\phi)$] and the synchronization phase [peak position of $P(\phi)$] is shifted, Fig. 3.1(b).

These steady-state properties do not provide any information on the underlying synchronization dynamics, especially if we are trying to discover possible underdamped and quantum-coherent phase dynamics. To test for these regimes, we now derive an effective quantum model.

### 3.3 Effective quantum model

In the synchronized regime, the classical equation of motion for $\langle \hat{b} \rangle = \beta = Re^{i\phi}$,

$$\dot{\beta} = i\Delta \beta + \frac{\gamma_1}{2} \beta - \gamma_2 |\beta|^2 \beta - F,$$  

has a stable fixed point $\beta_{ss} = R_{ss} e^{i\phi_{ss}}$. We linearize Eq. (3.1) around $\beta_{ss}$ by defining $\hat{b} = \beta_{ss} + \delta \hat{b}$, where $\delta \hat{b}$ describes fluctuations around $\beta_{ss}$. Neglecting terms of order $\mathcal{O}(\delta \hat{b}^3)$ and higher, we obtain

$$\dot{\hat{\rho}}_{\text{eff}} = -i \left[ \hat{H}_{\text{eff}}, \hat{\rho}_{\text{eff}} \right] + \gamma_1 \hat{\rho}_{\text{eff}} \hat{\rho}_{\text{eff}} + 4\gamma_2 |\beta_{ss}|^2 \hat{\rho}_{\text{eff}},$$  

with the effective Hamiltonian

$$\hat{H}_{\text{eff}} = -\Delta \delta \hat{b} \delta \hat{b} - i \frac{\gamma_2}{2} \left( \beta_{ss}^2 \delta \hat{b} \delta \hat{b} - \beta_{ss}^* \delta \hat{b}^* \delta \hat{b}^* \right).$$  

This effective model captures the major features of the full quantum model and thus allows at least qualitative predictions about the behaviour of the system, while quantitative agreement varies with parameters. A comparison of the outcomes of Eqs. (3.1) and (3.3) can be found in Sec. 3.8.3. The effective model is a squeezing Hamiltonian where the amount of squeezing depends on the classical steady-state solution $\beta_{ss}$.

Diagonalizing Eq. (3.4) leads to

$$\hat{H}_{\text{diag}} = -\Omega_{\text{eff}} \hat{c}^\dagger \hat{c} + \text{const.}.$$
Figure 3.2: Classical phase diagram and squeezing. (a) Overview of the classical synchronization regimes with sketches of typical phase-space trajectories. (b) Asymmetry of the steady-state squeezing ellipses, \( \max(\lambda_{\text{cov}}) / \min(\lambda_{\text{cov}}) \), obtained from the effective model. At the black crosses we show the squeezing ellipses (not to scale) with their radial direction aligned along \( \hat{e}_1 \). Two cuts at different forcing are shown below the figure. Parameters: \( \gamma_2 / \gamma_1 = 0.1 \). Note: Quantum noise leads to a finite threshold for synchronization [16, 32]. Below this threshold, the effective model is not applicable since classically a stable fixed point exists, but the quantum system settles on a LC. To improve the readability of (b) we nevertheless also plot the region where the effective model fails which here is only the case for very small force and detuning. (This figure has previously been published in [21].)

Here, \( \delta \hat{b} e^{-i\theta/2} = \cosh(\chi) \hat{c} + \sinh(\chi) \hat{c}^\dagger \), \( Ae^{i\theta} := -i \gamma_2 \beta_{ss}^2 / 2 \), \( \tanh(2\chi) = 2A/\Delta \), and \( \Omega_{\text{eff}} = \sqrt{\Delta^2 - 4A^2} \) is the effective frequency. The corresponding master equation reads

\[
\dot{\hat{\rho}}_{\text{diag}} = -i [\hat{H}_{\text{diag}}, \hat{\rho}_{\text{diag}}] + \Gamma_1 [\mathcal{D}[\hat{c}^\dagger] \hat{\rho}_{\text{diag}}] + \Gamma_1 [\mathcal{D}[\hat{c}] \hat{\rho}_{\text{diag}}], \tag{3.6}
\]

with \( \Gamma_1 = 4\gamma_2 |\beta_{ss}|^2 \sinh^2(\chi) + \gamma_1 \cosh^2(\chi), \Gamma_1 = 4\gamma_2 |\beta_{ss}|^2 \cosh^2(\chi) + \gamma_1 \sinh^2(\chi) \), and neglecting fast rotating terms, such as \( \hat{c} \hat{c} \hat{\rho}_{\text{eff}} \). The diagonalized, effective model is a damped harmonic oscillator with frequency \( \Omega_{\text{eff}} \) and damping \( \Gamma = \Gamma_1 - \Gamma_1 \). This unambiguously allows us to identify an underdamped phase dynamics regime following the standard procedure for a harmonic oscillator, i.e., we require \( \Delta^2 > 4A^2 \), which leads to a real-valued effective frequency \( \Omega_{\text{eff}} \). This is consistent with the corresponding classical dynamics derived from Eq. (3.3),

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leading to a second-order differential equation of the phase,

\[ \delta \ddot{\phi} + \Gamma \delta \dot{\phi} + \Omega^2 \delta \phi = 0. \]  

(3.7)

Here, \( \delta \phi = \phi - \phi_{ss} \) describes phase deviations from the steady-state phase \( \phi_{ss} \) and \( \Omega = \sqrt{\Delta^2 + (\gamma_2 R_{ss}^2 - \gamma_1/2)(3\gamma_2 R_{ss}^2 - \gamma_1/2)} \) is the bare frequency which is related to the effective frequency \( \Omega_{\text{eff}} = \sqrt{\Omega^2 - \Gamma^2/4} = \sqrt{\Delta^2 - \gamma_2^2 |\beta_{ss}|^4}; \) cf. [187] and Sec. 3.8.2. Before we discuss results from our effective quantum model, we briefly review the relevant features of the corresponding classical “phase diagram” of synchronization, Fig. 3.2(a). This phase diagram and its quantum analogue will help us to identify the parameter regime of underdamped phase motion, where we then can check for quantum coherence. We obtain the boundaries between the regimes of the classical phase dynamics from a linear stability analysis of Eq. (3.2); cf. [12] and Sec. 3.8.1. Notably, we distinguish two qualitatively different transitions from synchronization to no synchronization: At small forces, a saddle-node bifurcation characterizes the transition from synchronized (overdamped) dynamics directly to the LC regime. At larger forces, a regime of underdamped phase motion opens up before a Hopf bifurcation marks the onset of a LC which does not necessarily encircle the origin. Since we are actually interested in a quantum regime, it is worthwhile to see that these two qualitatively different transitions have also important consequences for the quantum dynamics. In particular, we find a qualitative change of behaviour in the squeezing properties of the steady state. Since \( \hat{H}_{\text{eff}} \) is quadratic in \( \delta \hat{b} \) the system is fully characterized by its covariance matrix \( \sigma_{ij} = \text{Tr}[\hat{\rho}_{\text{eff}}(\hat{X}_i, \hat{X}_j)/2] \) with the quadratures \( \hat{X}_1 = (\delta \hat{b} + \delta \hat{b}^\dagger)/\sqrt{2} \) and \( \hat{X}_2 = -i(\delta \hat{b} - \delta \hat{b}^\dagger)/\sqrt{2} \). The eigenvalues \( \lambda_{\text{cov}} \) of the covariance matrix determine the shape of the squeezing ellipse [83]. Their ratio (the asymmetry of the ellipses) is shown in Fig. 3.2(b). Notably, at small forces, it increases with larger detuning. In contrast, at larger forces where we predict underdamped phase dynamics, the ellipses become more circular while increasing the detuning \( \Delta \). Thus, the squeezing behaviour can be used as an indicator for the existence of a quantum regime of underdamped phase motion. The effective model becomes unstable if \( \Gamma = 0 \), which corresponds to the classical fixed point losing its stability. Additional details on squeezing can be found in Sec. 3.8.4.

### 3.4 Quantum coherence

Studying the effective model, we have identified the quantum regime of underdamped phase motion. Now we demonstrate that within this regime, it is possible to preserve quantum coherence for a significant time. To this end, we choose an initial state which possesses negativities in its Wigner function, Fig. 3.3(a), and show that these negativities persist for a long time compared to the characteristic timescale of the dynamics \( \Omega_{\text{eff}}^{-1} \). The dynamics due to Eq. (3.1) leads to a rotation of the state around the classical steady state \( \beta_{ss} \), Fig. 3.3(b). Notably, this dynamical evolution has little influence on the coherence and the negativities of the Wigner density survive many oscillations of the system, Fig. 3.3(c). After the loss of coherence, the
3.4. Quantum coherence

Figure 3.3: Quantum coherence. Wigner densities $W(x, p, t)$ of (a) the initial state $|\Psi(t = 0)\rangle \sim (|\beta_{ss} + 2\rangle + |\beta_{ss} - 2\rangle)$ and (b) at a later time. The underdamped phase dynamics rotates the state around the classical steady-state solution (yellow cross). (c) Wigner density $W(p, x = 0, t)$ with negativities that remain visible for many oscillations. Parameters: $\gamma_2/\gamma_1 = 0.1$, $F/\gamma_1 = 1.5 \times 10^3$ and $\Delta/\gamma_1 = 7 \times 10^2$. (a)-(c) show numerical solutions to the full model Eq. (3.1). (This figure is part of a figure previously published in [21].)

State remains in a classical mixture of two displaced states and settles into the steady state only on an even longer timescale; see Sec. 3.8.5 for a complete overview. All Wigner densities in Figs. 3.3(a)-3.3(c) are obtained by numerically solving the full master equation (3.1).

This behaviour is successfully predicted by our effective model, which allows us to quantify quantum coherence within the underdamped regime and eventually identify a quantum-coherent regime. The time scale on which the quantum system approaches the steady state is approximately given by the damping $\Gamma$. Thus, a necessary condition to observe quantum-coherent motion is $\Omega_{\text{eff}} > \Gamma$. Approaching the classical Hopf bifurcation, the damping $\Gamma$ becomes arbitrarily small. However, a small damping does not imply a small dephasing rate $\Gamma_{\text{deph}} = \Gamma_\uparrow + \Gamma_\downarrow$. The dephasing rate ultimately determines the lifetime of negativities, i.e., quantum coherence. With $\Gamma_\uparrow = \Gamma n_{\text{eff}}$ and $\Gamma_\downarrow = \Gamma (n_{\text{eff}} + 1)$, the dephasing rate $\Gamma_{\text{deph}}$ depends on both the damping $\Gamma$ and the effective occupation of the VdP oscillator $n_{\text{eff}}$. This effective occupation comes about due to the driven-dissipative character of the quantum oscillator even at zero environmental temperature, also called quantum heating [190]. It increases towards the boundaries of the underdamped regime, Fig. 3.4(a), counteracting the decreasing damping. Additional insight is obtained by identifying a quality factor for quantum coherence,
3. Quantum-coherent phase oscillations in synchronization

Figure 3.4: Quantum coherence. (a) Effective temperature in the underdamped regime, indicated by $n_{\text{eff}}$. The left white area corresponds to the overdamped regime. (b) Quality factor $\Omega_{\text{eff}}/\Gamma_{\text{deph}}$ in the underdamped regime. (c) and (d) show the effective oscillation frequency $\Omega_{\text{eff}}$ (dashed blue), damping $\Gamma$ (dash-dotted green), and dephasing rate $\Gamma_{\text{deph}}$ (red) as a function of the detuning $\Delta$. (c) At small force $F/\gamma_1 = 1.5$ the dephasing remains the dominant rate. (d) In contrast, at larger force $F/\gamma_1 = 10^3$ the frequency $\Omega_{\text{eff}}$ can significantly exceed both the dephasing and the damping (quantum-coherent regime). Parameters: $\gamma_2/\gamma_1 = 0.1$. (a)-(d) show the rates obtained from our effective quantum model which characterize the behaviour of the full system. (This figure is part of a figure previously published in [21].)

$\Omega_{\text{eff}}/\Gamma_{\text{deph}}$, which determines the lifetime of negativities in the Wigner density. Close to the instability and, more importantly, at large forcing and detuning, $\Omega_{\text{eff}}/\Gamma_{\text{deph}}$ increases and can become significantly larger than 1, Fig. 3.4(b). This is the quantum-coherent regime where negativities of the Wigner density can survive many oscillations of the system, Fig. 3.3(c). Regarding Fig. 3.4(b), the only remaining dimensionless parameter (apart from the normalized force and detuning) is the ratio of the damping rates $\gamma_2/\gamma_1$. It influences the region of stability of the effective model. For instance, increasing $\gamma_2/\gamma_1$ shifts the instability ($\Gamma = 0$) to larger detuning. This allows to achieve a comparable quality factor $\Omega_{\text{eff}}/\Gamma_{\text{deph}}$ at smaller forcing but similar detuning - mainly because $\Omega_{\text{eff}}$ increases with $\Delta$. In Figs. 3.4(c) and (d) we show all relevant rates in the underdamped regime at small and large forcings, respectively. In both cases $\Omega_{\text{eff}}$ increases, while $\Gamma$ and $\Gamma_{\text{deph}}$ decrease with larger detuning. At small force, Fig. 3.4(c), the dephasing rate remains the largest rate in the entire underdamped regime. Notably, for large $F$, Fig. 3.4(d), we find that $\Omega_{\text{eff}}$ can become significantly larger than both $\Gamma$ and $\Gamma_{\text{deph}}$, thus entering the quantum-coherent regime. This is the key element to observing long-lived quantum coherence.
To shed more light on the possibility to experimentally observe the transition from overdamped to underdamped synchronization dynamics, we investigate the spectrum $S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \hat{\mathbf{b}}(t)^\dagger \hat{\mathbf{b}}(0) \rangle$. We obtain $S(\omega)$ from the steady state of Eq. (3.1) by applying the quantum regression theorem or analytically from the effective model; see Sec. 3.8.6. The spectrum carries information on the frequencies of the driven VdP oscillator. Figure 3.5(a) shows $S(\omega)$ for a fixed external force and various detunings, corresponding to the overdamped (upper black spectrum) and underdamped (middle blue, lower red spectra) regime. In the overdamped regime the spectrum shows a single peak close to $\omega = 0$, indicating synchronization to the external force. With increasing detuning, the spectrum develops from a single peak to two peaks which now sit at approximately $\pm \Omega_{\text{eff}}$. A small remainder of the central peak at $\omega = 0$ becomes visible for a larger splitting of the main peaks. The emerging double peaks clearly indicate the transition from overdamped to underdamped phase dynamics, Fig. 3.5(b). The increasing asymmetry of $S(\omega)$ results from the coupling of amplitude and phase dynamics.
For even larger detuning, synchronization is lost which ultimately leads to a single peak in the spectrum at $\omega = \Delta$. A recent experiment synchronized two nanomechanical oscillators by coupling to a common cavity mode [40]. Curiously, the cavity output spectrum showed sidebands next to the common frequency of the locked oscillators. These sidebands were suggested to arise from (classical) underdamped phase motion of the oscillators, which is also consistent with the classical limit of our theory.

Interestingly, we find that the phase can even undergo self-oscillations. In the quantum regime, these phase self-oscillations appear (in analogy to the classical scenario) at the boundary of underdamped phase motion just before the loss of synchronization occurs. A circular LC opens up around the former stable fixed point. In the quantum regime this is smeared by quantum fluctuations and becomes visible only once the LC is large enough. If that LC expands even further, it will eventually come to resemble the original unsynchronized state: The LC encircles the origin of phase space and the corresponding phase distribution is flat, Fig. 3.1(a). However, in Fig. 3.5(c), this is not yet the case, i.e., the LC does not encircle the origin. The oscillator has still a tendency to be locked to the phase of the external force. This is also reflected in the corresponding phase distribution $P(\phi)$ which becomes asymmetric and shows the onset of a double peak structure. Notably, phase self-oscillations are accompanied by the appearance of a series of peaks in the spectrum, Fig. 3.5(d), representing higher harmonics of the main phase-oscillation frequency.

### 3.6 Experimental realization

The regime of underdamped quantum phase motion and even quantum phase self-oscillations could be experimentally studied in a variety of systems. For instance, trapped ions are promising candidate systems for studying synchronization in the quantum regime [15, 33]. The possibility to prepare nonclassical states experimentally [191] allows for probing the quantum-coherent nature of the underdamped phase dynamics. Based on parameters for trapped $^{171}$Yb$^+$ ions [15, 192, 193], we estimate that it should be possible to observe significant quantum coherence. In this scenario, the negative and nonlinear damping are both of the order of kHz, with $\gamma_2/\gamma_1 \sim 1$. To observe quantum-coherent underdamped phase dynamics the detuning $\Delta$ and the external force $F$ should be a few hundred kHz each. This is realistic, with frequencies of the motional state in the MHz regime. Furthermore, mechanical self-oscillations in cavity optomechanics have been discussed theoretically [123] and observed experimentally [121, 124]. Thus, they are well-suited to study synchronization, and classical synchronization phenomena have already been demonstrated experimentally [38–40, 42]. Yet another possible platform to observe quantum-coherent phase motion are superconducting microwave circuits. These are exceptional and highly tuneable platforms for experimentally investigating quantum systems. In principle, arbitrary quantum states can be realized [194–196]. Even, the faithful engineering of two-photon losses in such systems has been demonstrated [197]. This makes them very interesting for studying quantum-coherent phase motion and phase self-oscillations of a quantum VdP oscillator.
3.7 Conclusion

We have shown that the phase of a synchronized quantum Van der Pol oscillator exhibits intriguing underdamped and even quantum-coherent phase dynamics around the synchronized steady state. In order to explore this interesting regime, we have developed an effective quantum model and identified where the dephasing rate becomes sufficiently small to observe quantum-coherent phase motion. As a direct consequence, we have shown that this preserves a nonclassical quantum state for many phase oscillations. We estimate that this could readily be observed in state-of-the-art experiments. While we have analyzed the simplest synchronization phenomenon, to an external drive, the regime identified here will also show up in the quantum phase dynamics of two coupled oscillators or even lattices [32]. In the latter case, phenomena such as quantum motion of phase vortices may potentially become observable.

3.8 Supplemental material

3.8.1 Details on the classical synchronization phase diagram

Here, we discuss some additional details of the classical phase diagram of a Van der Pol (VdP) oscillator synchronizing to an external force, cf. Fig. 3.2(a). The boundaries for the regimes of overdamped, underdamped, and limit-cycle motion are obtained from a linear stability analysis of Eq. (3.2) (see for instance also Ref. [12] for more details). Using \( \beta = \beta_{ss} + \delta \beta \) and keeping only first order terms of \( \delta \beta \), the linearized equation of motion is

\[
\delta \dot{\beta} = i \Delta \delta \beta - \gamma_{2} \beta_{ss}^{2} \delta \beta^{*} + \frac{\gamma_{1}}{2} \delta \beta - 2 \gamma_{2} |\beta_{ss}|^{2} \delta \beta.
\]  

(3.8)

The eigenvalues \( \lambda \) of the corresponding Jacobi matrix are related to damping and effective frequency of the VdP oscillator and contain information about the properties of the corresponding fixed point \( \beta_{ss} \). In Fig. 3.6, we show the real and imaginary part of the eigenvalues \( \lambda_{1/2} = -2|\beta_{ss}|^{2} \gamma_{2} + \gamma_{1}/2 \pm \sqrt{\gamma_{1}^{4} \gamma_{2}^{2} - \Delta^{2}}. \) Note that, depending on parameters, Eq. (3.8) features either one or three fixed points. However, if there are three fixed points, see Fig. 3.6(a), only one of them is stable. We show the real and imaginary parts of the eigenvalues of a stable fixed point as solid lines and the eigenvalues corresponding to unstable fixed points as dashed lines. Note that in the synchronized regimes (both overdamped and underdamped) synchronization towards the stable fixed point occurs.

The phase diagram, cf. Fig. 3.2(a), shows that the regime of limit-cycle motion can be entered via two different types of bifurcations which we show here in detail: In Fig. 3.6(a) the regime of limit-cycle motion is entered via a saddle-node bifurcation, while in Fig. 3.6(b) a Hopf bifurcation marks the onset of limit-cycle oscillations. In both cases a single, unstable fixed point exists in the limit-cycle regime, i.e., the corresponding real parts of the eigenvalues, associated to damping, are positive. This indicates amplification and the stabilizing,
nonlinear effects that lead to a limit cycle are not included in the linear stability analysis. At smaller detuning, before the limit-cycle regime, synchronization towards a stable fixed point occurs. The important difference between the two possible transitions is determined by the imaginary part of the eigenvalues of this stable fixed point. The imaginary part is related to the oscillation frequency and approaching the saddle-node bifurcation, the imaginary parts remain zero. This implies that, for detunings below the bifurcation, no characteristic oscillation frequency exists. The non-zero imaginary parts visible in Fig. 3.6(a) belong to unstable fixed points, which are not of importance in the context of synchronization. In contrast, at larger force, Fig. 3.6(b), a Belyakov-Devaney transition [198] occurs even before the Hopf bifurcation. There the real parts of the eigenvalues become equal but remain negative (i.e. there exists a stable fixed point), yet the imaginary parts become nonzero. This implies that in this case the steady state is approached in an oscillatory fashion and determines the regime of underdamped phase motion. Only at even larger detuning a Hopf bifurcation occurs where the real parts become positive as well and a limit cycle is created.

The boundaries between the different regimes of classical synchronization phase dynam-
ics are determined by the following explicit expressions: (i) a saddle-node bifurcation given by $F^2 = \frac{2(\gamma_1 + \sqrt{\gamma_1^2 - 12\Delta^2})(-\gamma_1 \sqrt{\gamma_1^2 - 12\Delta^2} + 12\Delta^2 + \gamma_1^2)}{108\gamma_2^2}$, (ii) a transition from a stable node to a stable focus (Belyakov-Devaney transition) which is defined by $F^2 = \left(-\frac{\gamma_1}{2} + \Delta\right)^2 \frac{\Delta}{\gamma_2} + \frac{\Delta^2}{\gamma_2}$ with $|\Delta| > \gamma_1/4$, and (iii) a Hopf bifurcation described by $F^2 = \frac{1}{4} \frac{\gamma_1}{\gamma_2} \left(\alpha^2 + \frac{1}{64\gamma_2}\right)$ with $|\Delta| > \gamma_1/4$. Note that this linear analysis does not allow us to distinguish between stable self-oscillations of the phase (limit cycles not evolving around the origin) and ordinary limit cycles where the phase is monotonously increasing.

### 3.8.2 Classical dynamics of the effective quantum model

The effective quantum model, Eq. (3.3), allows us to discuss the corresponding classical dynamics which is given by $\delta \dot{\beta} = \mathcal{H} \delta \beta$ where $\delta \beta = \beta - \beta_{\text{st}}$, $\mathcal{H} \delta \beta = -\gamma_1 \delta \beta - \gamma_2 \delta \beta^\ast + \frac{\gamma_1}{\gamma_2} \delta \beta - 2\gamma_2 |\beta_{\text{st}}| \delta \beta$. This is equivalent to the linearized equation (3.8) confirming that we have indeed derived the correct linearized quantum model. It is instructive to first split the complex amplitude $\beta$ into amplitude $R$ and phase $\phi$ such that $\beta = R e^{i\phi}$, and then obtain the corresponding equations for the amplitude and phase deviations $\delta R$ and $\delta \phi$. These deviations are simply defined as the difference between the actual amplitude $R$ (phase $\phi$) from the steady-state amplitude $R_{\text{st}}$ (steady-state phase $\phi_{\text{st}}$), i.e. $\delta R = R - R_{\text{st}}$ and $\delta \phi = \phi - \phi_{\text{st}}$. Since $\delta R$ and $\delta \phi$ are small, $\delta R$ is approximately the change in direction of $R_{\text{st}}$ and $\delta \phi$ is approximately the change perpendicular to this. For $\delta \beta = r e^{i\phi}$ we then obtain $\delta \phi \approx r \sin(\phi_{\text{st}} - \phi)$ and $\delta R \approx r \cos(\phi_{\text{st}} - \phi)$ and with this

$$\delta \phi = \Delta \delta R - \left(\gamma_2 R_{\text{st}}^2 - \frac{\gamma_1}{2}\right) \delta \phi, \quad \delta R = - \left(3\gamma_2 R_{\text{st}}^2 - \frac{\gamma_1}{2}\right) \delta R - \Delta \delta \phi,$$

(3.9)

(3.10)

which can be combined to a second-order differential equation for the phase,

$$\ddot{\delta \phi} + \Gamma \dot{\delta \phi} + \Omega^2 \delta \phi = 0.$$

(3.11)

Here we have defined $\Gamma = (4\gamma_2 R_{\text{st}}^2 - \gamma_1)$ and $\Omega = \sqrt{\Delta^2 + \left(\gamma_2 R_{\text{st}}^2 - \frac{\gamma_1}{2}\right)\left(3\gamma_2 R_{\text{st}}^2 - \frac{\gamma_1}{2}\right)}$. Notably Eq. (3.11) is a common harmonic oscillator which allows for overdamped as well as underdamped motion. The transition from overdamped to underdamped solutions is characterized by $\Omega^2 = \Gamma^2/4$, i.e. where the effective oscillation frequency of the system $\Omega_{\text{eff}} = \sqrt{\Omega^2 - \Gamma^2/4}$ becomes real-valued. The solution to Eq. (3.11) becomes unstable if $\Gamma < 0$, revealing the onset of limit-cycle motion. The limit-cycle motion itself depends on nonlinear effects to stabilize and thus cannot be described with the linearized equations.

The parameters $\Gamma$ and $\Omega_{\text{eff}}$ obtained from this classical analysis are equal to the damping and effective frequency appearing in the effective quantum model.
3. Quantum-coherent phase oscillations in synchronization

Figure 3.7: Full and effective quantum model. Steady-state Wigner densities of (a) the full and (b) of the effective quantum model. The corresponding “phase trajectories” in (c) show similar oscillating behaviour, although relaxing to a different steady state. The black, dashed line gives the classical trajectory for comparison. In (d) we show the variance $\text{Var}(\hat{r}_\perp)$ as a function of detuning. The deviations of the effective from the full model increase towards the Hopf bifurcation, where the effective model breaks down. Parameters: $\gamma_2/\gamma_1 = 0.1$, $F/\gamma_1 = 4$, and $\Delta/\gamma_1 = 1.8$. (This figure has previously been published in the Supplemental material of [21].)
3.8.3 Comparison of the full and the effective quantum model

Here we compare results from the full quantum model, Eq. (3.1), to results from the effective model, Eq. (3.3), and the outcome of the classical equations (3.9) and (3.10). In Fig. 3.7(a) and (b) we show the steady-state Wigner density obtained from the full quantum model and the effective model respectively. The result of the effective model needs to be displaced to the classical steady state $\beta_{ss}$, indicated by the white cross. The Wigner densities obtained from the full and the effective quantum model match reasonably well. The parameters were chosen such that first deviations become visible: (i) The Wigner density of the full model is no longer centered exactly around the classical solution $\beta_{ss}$, while the effective model does so by construction. (ii) The effective quantum model is described by a squeezing Hamiltonian, cf. Sec. 3.3. Thus the corresponding Wigner densities are ellipses, while the full model can lead to additional curvature in the Wigner density (more banana-shaped).

Within the effective model synchronization attracts the system’s dynamics towards the stable fixed point $\beta_{ss}$. We can capture the dynamics using small deviations around $\beta_{ss}$. A natural choice are deviations in radial direction, $\delta R$, and in phase direction, $\delta \phi$, similar to the classical treatment. We can define corresponding operators $\hat{r} = \cos(\phi_{ss}) \hat{x} / x_{zpf} + \sin(\phi_{ss}) \hat{p} / p_{zpf}$ in radial direction and perpendicular to it, $\hat{r}_\perp = \sin(\phi_{ss}) \hat{x} / x_{zpf} - \cos(\phi_{ss}) \hat{p} / p_{zpf}$. With this, deviations of the phase can be approximated via $\delta \phi \approx -\langle \hat{r}_\perp \rangle / R_{ss}$ such that the full phase is given by $\phi(t) \approx \phi_{ss} - \langle \hat{r}_\perp \rangle(t) / R_{ss}$. We show the phase as a function of time in Fig. 3.7(c). The system shows underdamped phase motion, i.e. a few damped oscillations can be observed in the full and effective quantum model, as well as in the classical simulation. It is consistent with the corresponding Wigner densities, that the trajectories of the full and effective quantum model are damped towards a different steady state. Only the steady state of the effective quantum model and the classical equations are equal by construction. Note that the relation of $\langle \hat{r}_\perp \rangle$ to the phase deviations $\delta \phi$ is only accurate if the deviations are small. In Fig. 3.7(d) we show the variance $\text{Var}(\hat{r}_\perp)$ as a function of detuning. For small $\Delta$ synchronization works best, i.e. the Wigner density is more confined in phase space and thus the resulting variance is small. Deviations between the full and effective quantum model appear with increasing detuning. Then, synchronization becomes weaker and the full model can develop a less ellipse-like Wigner density. Approaching the Hopf bifurcation the variance within the effective model blows up, signalling the break-down of the model. The full model shows an increasing variance, which is consistent with the synchronization becoming weaker and the Wigner density becoming more smeared out.

3.8.4 Details on the squeezing

In Sec. 3.3 we derived the squeezing Hamiltonian of our effective model, Eq. (3.4), and discussed the asymmetry of the squeezing ellipses in Fig. 3.2(b). Due to the quadratic Hamiltonian, the state is fully characterized by its covariance matrix $\sigma_{ij} = \text{Tr}[\hat{\rho}_{\text{eff}}(\hat{X}_i, \hat{X}_j)/2]$ with the quadratures $\hat{X}_1 = (\hat{\delta b} + \hat{\delta b}^\dagger) / \sqrt{2}$ and $\hat{X}_2 = -i(\hat{\delta b} - \hat{\delta b}^\dagger) / \sqrt{2}$ (same definition as in Sec. 3.3).
The equation of motion for the covariance matrix can be expressed in the following form,

\[ \dot{\sigma} = M \sigma + \sigma M^\top + D, \]  

(3.12)

with the matrices

\[ M = \begin{pmatrix} i(r - r^*) + \gamma_1/2 - 2\gamma_2|\beta_{ss}|^2 & r + r^* - \Delta \\ r + r^* + \Delta & -i(r - r^*) + \gamma_1/2 - 2\gamma_2|\beta_{ss}|^2 \end{pmatrix} \]  

(3.13)

\[ D = \begin{pmatrix} \gamma_1/2 + 2\gamma_2|\beta_{ss}|^2 & \gamma_1/2 + 2\gamma_2|\beta_{ss}|^2 \end{pmatrix}. \]  

(3.14)

Here we used \( r = i\gamma_2\beta_{ss}^2/2 \) for brevity. The steady-state solution to this equation of motion, \( \dot{\sigma} = 0 \), can be analytically obtained, resulting in a 2 x 2 matrix \( \sigma \) that depends only on system parameters and the classical steady-state amplitude \( \beta_{ss} \). Then, the eigenvalues of the covariance matrix \( \sigma \) can be calculated and analyzed. The ratio of these eigenvalues determines the asymmetry of the squeezing ellipses discussed in Sec. 3.3. However, also the absolute amount of squeezing can be analyzed by comparing to the size of the vacuum state. If any direction of the squeezing ellipse becomes smaller than the width of the vacuum state this is referred...
to as squeezing below shot noise. To this end, we calculate the shot-noise covariance matrix
\[ \sigma_{sn} = \text{Tr}(|0\rangle\langle 0| \{\hat{X}_i, \hat{X}_j\}/2) \], which is diagonal and has \( \lambda_{sn} = 1/2 \) as doubly degenerate eigenvalue. We compare this to the smallest eigenvalue \( \min(\lambda_{\text{cov}}) \) of the covariance matrix of the synchronized VdP oscillator. Squeezing below shot noise occurs for values \( \min(\lambda_{\text{cov}}) < 1/2 \).

Interestingly, the synchronized VdP oscillator does feature squeezing below shot noise at small detuning and sufficiently large forcing. As shown in Fig. 3.8, for a fixed detuning, squeezing becomes stronger if the external force \( F \) is increased, eventually dropping below the shot noise value. In combination with Fig. 3.2(b), we conclude that approximately the radial direction is squeezed. Since squeezing below the shot noise level occurs mainly at small detuning, it occurs mostly within the regime of overdamped phase motion, but can reach into the underdamped regime as well. However, approaching the classical Hopf bifurcation, i.e., the instability of the effective model, by increasing the detuning \( \Delta \), the squeezing necessarily decreases since the Wigner density smoothly transforms back into a (circular) limit cycle.

This parameter dependence of the absolute squeezing can also be directly explained from the squeezing Hamiltonian, Eq. (3.4). At first sight squeezing depends on the steady state of the system, i.e., \( \beta_{ss} \), and thus has an intricate dependence on all parameters. However, we generally observe that large forcing \( F \) leads to large values of \( |\beta_{ss}| \). Notably, the squeezing Hamiltonian does not depend on this absolute value, but on the complex value \( \beta_{ss}^2 \) instead. Investigating the steady state Wigner density of the synchronized VdP oscillator, we observed in Fig. 3.1(b) a crucial dependence on the detuning: Although the value \( |\beta_{ss}| \) does slightly decrease with \( \Delta \), the more important effect is a rotation in phase space, corresponding to a change of the synchronization phase. Thereby \( \beta_{ss} \) transforms from an almost real quantity to an almost purely imaginary quantity, thus significantly decreasing the real part of \( \beta_{ss}^2 \) even if its modulus would be conserved completely. Therefore, we can conclude that large squeezing appears if the force is sufficiently large compared to the detuning such that the synchronization phase (the peak position of the phase distribution) is close to the ideal value of 0 or \( \pi \).

### 3.8.5 Long time evolution of coherent synchronization dynamics

In Sec. 3.4 we discuss that the synchronization dynamics can preserve quantum coherence for a significant number of oscillations of the system. Here, in Fig. 3.9, we want to show how an initially prepared superposition state loses coherence and finally relaxes to the synchronized steady state. We numerically simulate the full quantum model to stress that this behaviour, expected due to a sufficiently small dephasing rate obtained from our effective quantum model, can indeed be observed (although quantitative deviations occur). The beginning of this time evolution is also shown and described in Fig. 3.3, but will be repeated here for completeness.

Starting with a superposition state, Fig. 3.9(a), the Wigner density shows interference fringes with negativities. The synchronization dynamics described by the full master equation (3.1), leads to rotations around the classical steady state (yellow cross), Fig. 3.9(b). The
3. Quantum-coherent phase oscillations in synchronization

Figure 3.9: Long time evolution of coherent synchronization dynamics. Wigner densities $W(x, p, t)$ of (a) an initial superposition state $|\Psi(t = 0)\rangle \sim (|\beta_{ss} + 2\rangle + |\beta_{ss} - 2\rangle)$, (b)-(d) several snapshots at later times, and (e) the final steady state. The underdamped synchronization phase dynamics rotates the state around the classical steady-state solution (yellow cross). The interference fringes vanish due to dephasing and a classical mixture of displaced states remains, still rotating around the classical steady state. The displaced states eventually merge to the steady state on a timescale set by the damping. (e) and (f) show cuts of the Wigner densities, i.e. $W(p, x = 0, t)$, as a function of time. These time evolutions clearly show that dephasing and relaxing into the steady state occur on separate timescales. Parameters as in Fig. 3.3: $\gamma_2/\gamma_1 = 0.1$, $F/\gamma_1 = 1.5 \times 10^3$, and $\Delta/\gamma_1 = 7 \times 10^2$. (This figure has previously been published in the Supplemental material of [21].)
3.8. Supplemental material

Figure 3.10: Full and effective spectrum. The spectrum $S(\omega)$ of a synchronized VdP oscillator obtained from the full master equation (3.1), and the corresponding $S_{\text{eff}}(\omega)$ calculated analytically from the effective quantum model (dashed green lines). The three curves show the spectrum for different detuning $\Delta/\gamma_1 = 0$ (black), $\Delta/\gamma_1 = 2 \times 10^4$ (blue), and $\Delta/\gamma_1 = 5 \times 10^4$ (red). The effective spectrum features two (possibly degenerate) peaks close to $\pm \Omega_{\text{eff}}$, and cannot capture the third peak appearing at larger detuning. Parameters as in Fig. 3.5(a). (This figure has previously been published in the Supplemental material of [21].)

3.8.6 Analytical spectrum

In the over- and underdamped regime we can also obtain the spectrum from the analytical solution to our effective model. We start from Eq. (3.4), the squeezing Hamiltonian, and write down the quantum Langevine equations,

$$
\delta \dot{b} = i \Delta \delta b - \frac{\Gamma}{2} \delta b - \frac{\gamma_2}{2} \beta_{ss}^2 \delta b^\dagger + \sqrt{\Gamma} \xi, \quad (3.15)
$$

$$
\delta b^\dagger = -i \Delta \delta b^\dagger - \frac{\Gamma}{2} \delta b^\dagger - \frac{\gamma_2}{2} \beta_{ss}^* \delta b + \sqrt{\Gamma} \xi^\dagger. \quad (3.16)
$$

Here the noise operators $\xi$ and $\xi^\dagger$ represent white noise, fulfilling $\langle \xi^\dagger(t) \xi(t') \rangle = \tilde{n} \delta(t - t')$ and $\langle \xi(t) \xi^\dagger(t') \rangle = (\tilde{n} + 1) \delta(t - t')$ and $\tilde{n} = 1/(4 \gamma_2 |\beta_{ss}|^2 / \gamma_1 - 1)$ is obtained from the dissipation rates of Eq. (3.3), i.e. we identified $\gamma_1 \equiv n \Gamma$ and $4 \gamma_2 |\beta_{ss}|^2 \equiv (n + 1) \Gamma$. Eqs. (3.15) and (3.16) are easily solved in Fourier space where the problem simplifies to finding the inverse of a $2 \times 2$-matrix. Choosing the convention $\delta b(\omega) = \int_{-\infty}^{+\infty} dt e^{i \omega t} \delta b(t)$ and $\delta b^\dagger(\omega) = \int_{-\infty}^{+\infty} dt e^{-i \omega t} \delta b^\dagger(t)$ we find
3. Quantum-coherent phase oscillations in synchronization

\[
\delta \hat{b}(\omega) = \frac{[-i(\omega - \Delta) + \Gamma/2] \sqrt{\Gamma}}{\Delta^2 - (\omega + i\Gamma/2)^2 - \gamma_2^2 |\beta_{ss}|^4} \hat{\xi}(\omega) - \frac{-\gamma_2 \beta_{ss}^2 \sqrt{\Gamma}}{\Delta^2 - (\omega + i\Gamma/2)^2 - \gamma_2^2 |\beta_{ss}|^4} \hat{\xi}^\dagger(-\omega),
\]

\[
\delta \hat{b}^\dagger(\omega) = \frac{-\gamma_2 \beta_{ss}^2 \sqrt{\Gamma}}{\Delta^2 - (\omega + i\Gamma/2)^2 - \gamma_2^2 |\beta_{ss}|^4} \hat{\xi}(\omega) + \frac{[-i(\omega + \Delta) + \Gamma/2] \sqrt{\Gamma}}{\Delta^2 - (\omega + i\Gamma/2)^2 - \gamma_2^2 |\beta_{ss}|^4} \hat{\xi}^\dagger(-\omega).
\]

Within the effective model the fluctuation spectrum \(S_{\text{eff}}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \delta \hat{b}^\dagger(t) \delta \hat{b}(0) \rangle = \int_{-\infty}^{+\infty} d\omega' \langle \delta \hat{b}^\dagger(-\omega) \delta \hat{b}(\omega') \rangle\) can be obtained from this solution by evaluating the relevant noise correlators. We find

\[
S_{\text{eff}}(\omega) = \frac{\Gamma \gamma_2^2 |\beta_{ss}|^4 + \bar{n} \Gamma [(\Gamma/2)^2 + \gamma_2^2 |\beta_{ss}|^4 + (\omega + \Delta)^2]}{[(\omega - \sqrt{\Delta^2 - \gamma^2 |\beta_{ss}|^4})^2 + (\Gamma/2)^2] \left( [\omega + \sqrt{\Delta^2 - \gamma^2 |\beta_{ss}|^4})^2 + (\Gamma/2)^2 \right)}.
\] (3.17)

This spectrum features peaks close to \(\omega = \pm \sqrt{\Delta^2 - \gamma^2 |\beta_{ss}|^4} = \Omega_{\text{eff}}\).
3.8. Supplemental material
CLASSICAL NONLINEAR DYNAMICS OF TOPOLOGICAL TRANSPORT IN BOSONIC MODELS

In this chapter we discuss first results from our ongoing work about classical nonlinear dynamics of topological insulators in bosonic systems. While the previous chapters of this thesis dealt with at most two coupled nonlinear oscillators, here we turn to investigate large lattices. We introduced topological insulators in Sec. 1.2 and want to start off this chapter by reviewing important nonlinear processes in order to motivate the combination of both. Several bosonic systems have been found to show topological edge states, e.g., photonic systems [66], ultracold atoms [199], magnons [200, 201], and various phononic systems [63–65, 67, 202, 203]. While studies based on single-particle Hamiltonians lead to similar results for bosonic and fermionic systems, fundamental differences arise when it comes to nonlinearities and interactions. Here we focus on the classical dynamics of bosonic systems. First examples of nonlinear bosonic systems with topologically protected edge states have been recently studied. We give an overview about these previous results and then introduce the half Bernevig-Hughes-Zhang (BHZ) model, which describes a Chern insulator with well-known properties. This is the linear lattice model featuring topological transport which we then investigate in the presence of a local Kerr-like nonlinear interaction. In particular, we present a linear stability analysis that gives insight into the unstable scattering processes out of the linear topological edge state enabled by the nonlinearity. Moreover, we study the time evolution of the nonlinear Chern insulator on a finite strip with periodic boundary conditions. We relate the observed behaviour after some intermediate time to our results obtained from the linear stability analysis. However, eventually the edge state becomes unstable and radiates into the bulk.
4. Classical nonlinear dynamics of topological transport in bosonic models

4.1 Nonlinear phenomena

Let us consider for a moment light traveling through a nonlinear medium [204]: In a nonlinear medium the response of the material to the light scales not linearly with the light amplitude, but with some higher power of the light amplitude, which defines the order of the nonlinearity. For example, a second-order nonlinearity leads to a quadratic response, while, in a Hamiltonian formulation, it is described by a combination of three operators. An important difference between linear and nonlinear media is that the former cannot change the frequency of the impinging light, but the latter does. A second-order nonlinearity enables three-wave mixing, i.e., it allows processes that involve three “waves” or already using a “quantized” description, three photons. One example of a three-wave mixing process is second harmonic generation which combines two photons of the input frequency $\omega$ to an output frequency $2\omega$. This is actually used commercially to, e.g., convert laser light to other spectral ranges. Sum- and difference-frequency generation are also three-wave mixing processes. They combine for example two incoming frequencies $\omega_1$ and $\omega_2$ to an output frequency $\omega_1 + \omega_2$ or convert an incoming frequency $\omega_1$ into a frequency $\omega_1 - \omega_2$ and a photon of frequency $\omega_2$ (to fulfill energy conservation). Note that difference frequency generation can take place via stimulated emission if both frequencies $\omega_1$ and $\omega_2$ are sent into the nonlinear material. In this case one speaks of parametric amplification because the incoming $\omega_2$ frequency is amplified in this process. On the other hand, in a process called parametric downconversion a single input frequency is sufficient and the output frequencies are produced via spontaneous emission.

In a rather similar fashion, higher-order nonlinearities lead to higher-order harmonic generation and higher-order frequency mixing. Later in this chapter, we will consider a third-order nonlinearity, which is often referred to as a Kerr nonlinearity. This is because, in addition to four-wave mixing, it also gives rise to another nonlinear effect (called the Kerr-effect): Depending on the intensity of the light inside the nonlinear medium its refractive index is changed. This can lead to self-focusing inside the nonlinear material (depending on the sign of the refractive index change), i.e., the light is focused towards regions where the intensity of light is higher. This is just one possible self-action effect, where the light modifies its own propagation by nonlinearly changing the material properties.

Notably, because even the linear response is not instantaneous, real materials give rise to dispersion, i.e., different frequencies travel through the medium at different speeds. This implies that a wavepacket sent through a medium (even a linear one) will be broadened due to its different frequency components. Most interestingly, a nonlinear medium allows to compensate for dispersion. Of course, this is not only the case for optical nonlinear systems: It is known that generally nonlinear Schrödinger equations, which can describe rather different physical situations, can feature so-called solitons [205]. Solitons are localized wavepackets that propagate through the nonlinear medium, preserving their shape. Furthermore, solitons remain unchanged (up to a phase) when interacting with other solitons. If this second criterion is not fulfilled, one speaks of a solitary wave instead. The origin of solitons is the cancellation of dispersion due to counteracting nonlinear effects. Solitons that appear as a
peak are called bright solitons, while so-called dark solitons appear as dips within some background. Both self-repulsive and self-focusing nonlinearities can in principle lead to solitonic solutions.

These examples of nonlinear behaviour give just a small insight into the amount of phenomena that can be explored beyond the linear regime. Topological insulators themselves represent a highly interesting, fast developing field of research [68], but they are most often discussed in the context of linear, i.e., noninteracting systems. In the following, we want to investigate how topological effects in bosonic systems are affected by nonlinearities. Interesting questions range from the fate of topological edge states in the presence of a nonlinearity, to the idea of nonlinearly changing the (local) topology. In particular, the emergence of solitons in topological insulators play a central role in the literature existing so far. We give a brief overview about those results in the next section.

4.2 Brief overview: Nonlinear topological insulators

Recently, the idea of combining bosonic topological insulators with nonlinearities lead to first theoretical studies. It has been shown that already weak nonlinearities can be used to generate interesting effects, e.g., they can give rise to novel topological states [166, 206, 207]. In these studies however, it is sufficient to use a linearized model, i.e., a general Bogoliubov Hamiltonian. This is a bilinear Hamiltonian (thus leading to linear equations of motion), which includes squeezing terms, that can, e.g., originate from weak nonlinear interactions in a Bose-Einstein condensate [206, 207] or can be generated using a nonlinear crystal or an optomechanical system [165]. It is even possible to make use of the thereby arising topological protection to engineer a non-reciprocal amplifier [165]. In this thesis, however, we are interested in truly nonlinear behaviour instead of effects that can be engineered via nonlinearities. Such behaviour has also been addressed in several recent publications. While the involved nonlinearity typically was Kerr-like, various different geometries (e.g. honeycomb [208, 209], Kagome [210], square [211], SSH [212] lattices) were investigated using different combinations of numerical and approximative methods.

Several studies are based on photonic Floquet topological insulators [208, 209, 211]. In Ref. [208] self-localized, rotating wavepackets in the bulk are found. This can be understood as a nonlinearity-induced local defect in the bulk that forms a small domain with different topology, thus creating an artificial edge within the bulk. The wavepacket at the same time creates this defect located in the bulk bandgap and populates the associated chiral edge state. The authors refer to these localized wavepackets as solitons, that appear for both attractive and repulsive nonlinearity. The solitons they find are long-lived and recreate their shape periodically, although they are finally unstable.

Localized nonlinear edge modes (at the real edge of the system) were found and studied in Ref. [209]. It was demonstrated that those unidirectional nonlinear edge modes profited from topological protection, while in the topological trivial case they were slowed down due to scattering. These results were found by describing the edge of the 2D system with a 1D
4. Classical nonlinear dynamics of topological transport in bosonic models

nonlinear Schrödinger equation. In Ref. [211], a numerical study based on a square lattice, edge solitons were found both in the topologically trivial and nontrivial phase. In particular, the topologically trivial phase leads to a topology-inducing traveling edge soliton, which can be explained similar to the solitons in [208]: The nonlinearity itself induces the conditions for a local edge state. In contrast, if the underlying linear model is topologically nontrivial, the nonlinear system gives rise to solitons of various powers which inherit some robustness against perturbations of the edge channel. This is not the case for the self-induced solitons, although both types can cope with corners and site-defects without backscattering.

Similar results were found in Ref. [212] for a nonlinear 1D system. There, a SSH array was investigated and with increasing strength of the nonlinearity a transition from the topologically trivial to a topological nontrivial phase was observed, resulting also in edge solitons. Another very recent study of a 1D bosonic dimer chain focuses on topological gap solitons and their behaviour in contrast to non-topological gap solitons [213].

Having implementations with exciton-polaritons in mind, Ref. [214] shows first the existence of a stationary topological nonlinear edge state, which, however, is unstable. This instability leads to the creation of topological quasi-solitons - long-lived localized traveling edge solitons that weakly radiate into the bulk. These quasi-solitons can travel around corners and defects and even along short passages where the linear model does not support an edge state. In [210] several different types of edge solitons are identified on a Kagome lattice and it is shown that they are not merely solitary waves, but true solitons in the sense that they don't scatter amongst themselves.

In Ref. [215] topological states in a nonlinear optical network forming a $Z_2$-topological insulator were investigated. There it was found that the topological edge states can be stable even within the nonlinear regime.

Note that generally the interactions in a Bose-Einstein condensates are described by the Gross-Pitaevskii equation which contains a cubic, Kerr-like nonlinear term. Several studies of topological properties in interacting Bose-Einstein condensates have been recently published [216, 217] and contribute to the knowledge about the interplay of topology and nonlinear behaviour. For example, Ref. [217] finds that an increased intensity (and thus stronger nonlinearity) can lead to a “topological inversion”, where the signs of the Chern numbers are reversed. This implies that the propagation direction of the edge channel is changed due to the nonlinear interaction.

4.3 The half BHZ model

In this section we introduce the linear topological insulator model that we investigate in the following sections with an additional nonlinear term. We aim for the simplest tight-binding model that has nontrivial topological properties. In order to keep the implementation simple, we want to consider only nearest-neighbour interactions on a square lattice. These criteria are met by the half BHZ (Bernevig-Hughes-Zhang) model [218]. This is a widely studied
4.3. The half BHZ model

An illustration of the coupling terms in Hamiltonian (4.1). The underlying square lattice has two orbitals per lattice site. There is a real-valued tunneling amplitude $B$ between same orbitals on neighbouring sites, while the spin-orbit-like interaction $A$ between different orbitals on neighbouring sites has either a real-valued amplitude (top-down interaction) or an imaginary-valued amplitude (left-right interaction). The sketched coordinate systems on the left illustrate our conventions for $x$- and $y$-direction as well as the indices $n$, $m$ in this chapter.

Figure 4.1: Half BHZ model. An illustration of the coupling terms in Hamiltonian (4.1). The underlying square lattice has two orbitals per lattice site. There is a real-valued tunneling amplitude $B$ between same orbitals on neighbouring sites, while the spin-orbit-like interaction $A$ between different orbitals on neighbouring sites has either a real-valued amplitude (top-down interaction) or an imaginary-valued amplitude (left-right interaction). The sketched coordinate systems on the left illustrate our conventions for $x$- and $y$-direction as well as the indices $n$, $m$ in this chapter.

model of a Chern insulator with well-known properties, which can be used as a reference for our results in the nonlinear regime.

First, consider a square lattice with two orbitals per site, labeled $a$- and $b$-orbital. Introducing nearest-neighbour hopping $B$ for both orbitals separately leads to two independent bands. Then, introducing an additional nontrivial hopping $A$ between the orbitals on neighbouring sites, leads to the Hamiltonian of the half BHZ model

\[
\hat{H}_{BHZ} = \frac{1}{2} \sum_{n,m} \left[ 2W \left( \hat{a}^\dagger_{n,m} \hat{a}_{n,m} - \hat{b}^\dagger_{n,m} \hat{b}_{n,m} \right) + B \left( \hat{a}^\dagger_{n+1,m} \hat{a}_{n,m} + \hat{a}^\dagger_{n+1,m} \hat{a}_{n,m} - \hat{b}^\dagger_{n,m+1} \hat{b}_{n,m} - \hat{b}^\dagger_{n,m+1} \hat{b}_{n,m} + \text{h.c.} \right) \right. \\
\left. + A \left( -i \left( \hat{a}^\dagger_{n+1,m} \hat{b}_{n,m} + \hat{b}^\dagger_{n+1,m} \hat{a}_{n,m} \right) - \left( \hat{a}^\dagger_{n+1,m} \hat{b}_{n,m} - \hat{b}^\dagger_{n+1,m} \hat{a}_{n,m} \right) + \text{h.c.} \right) \right].
\]  

(4.1)
Here, $W$ determines the energy difference between the two orbitals (also called mass term) and $m, n$ label the sites in $x$- and $y$-direction respectively, see Fig. 4.1 for an illustration of the coupling terms and direction conventions. Originally, this model was put forward in the context of semiconductor physics, where the coupling $A$ originates from spin-orbit coupling for a fixed spin. Here, we focus on the picture of a bosonic lattice with two orbitals per unit cell and $\hat{a}_{nm}$ and $\hat{b}_{nm}$ are bosonic annihilation operators for the $a$-orbital or $b$-orbital, respectively.

Note that the half BHZ model, Eq. (4.1), breaks time-reversal symmetry and describes a Chern insulator. In contrast, the (full) BHZ model basically doubles the Hilbert space (by including both “spin up” and “spin down”) to preserve time-reversal symmetry instead, which gives rise to a $\mathbb{Z}_2$-topological insulator. A didactic introduction addressing both the half and the full BHZ model can be found, e.g., in Ref. [69].

To study the bulk properties of the half BHZ Hamiltonian (4.1), we assume translational symmetry in $x$- and $y$-direction and make a plane wave ansatz,

$$\hat{a}_{nm} = \frac{1}{\sqrt{V}} \sum_k e^{i\tilde{k}\tilde{r}} \hat{a}_{\tilde{k}}$$

$$\hat{b}_{nm} = \frac{1}{\sqrt{V}} \sum_k e^{i\tilde{k}\tilde{r}} \hat{b}_{\tilde{k}}.$$  (4.2)

Here, we used $\tilde{k} = (k_x, k_y)^T$ and $\tilde{r} = (m, n)^T$, where we set the lattice constant to $d = 1$, and $V$ denotes the volume. Introducing the ansatz (4.2) into the Hamiltonian (4.1), we only need to use $\sum_j e^{i(k-k')j} = \sqrt{V} \delta_{kk'}$ to express the bulk Hamiltonian as $\tilde{H} = \sum_{\tilde{k}} (\hat{a}_{\tilde{k}}^T \hat{b}_{\tilde{k}}^T) h(\tilde{k}) (\hat{a}_{\tilde{k}} \hat{b}_{\tilde{k}})^T$ with

$$h(\tilde{k}) = (W + B [\cos(k_x) + \cos(k_y)]) \sigma_z + A \sin(k_x) \sigma_x + A \sin(k_y) \sigma_y$$  (4.3)

and $\sigma_{x,y,z}$ the Pauli matrices. The eigenvalues of $h(\tilde{k})$ can be readily obtained as

$$E(k_x, k_y) = \pm \sqrt{(W + B [\cos(k_x) + \cos(k_y)])^2 + A^2 \sin^2(k_x) + A^2 \sin^2(k_y)},$$  (4.4)

and describe the bulk band structure. The band structure consists of two bands that are symmetrically arranged around $E = 0$, see Fig. 4.2(a). For most parameter values there exists a bandgap: Only if $W/B = 0$ or $W/B = \pm 2$ the bandgap closes and the bands touch for specific $k_x$ and $k_y$ values. Here, we focus on the parameters where indeed a bandgap exists and the system is clearly insulating.

The next step is to characterize the topological properties of the half BHZ model. Therefore, the Chern number $Q$ of the lower band is of particular interest and allows to distinguish the topologically trivial $Q = 0$ situation that does not feature edge states, from topologically nontrivial situations. For the half BHZ model one finds [69]

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1There are also trivial cases that require at least one coupling amplitude, $A$ and/or $B$, to be zero.
4.3. The half BHZ model

Figure 4.2: Edge states in the half BHZ model. (a) The bulk band structure of the half BHZ model featuring a bandgap. The Chern number of the lower band is $Q = -1$. Parameters: $W/B = -1.5, A/B = 1$. (b) Introducing edges to the system, i.e., calculating the band structure on a strip of, e.g., 30 sites in $y$-direction but still infinitely extended in $x$-direction, leads to edge states inside the bandgap. In (c) we show that these edge states are indeed localized at the two edges of the lattice, by showing the probability distribution along the lattice sites $|\langle \Psi_n | \Psi_{\text{edge}} \rangle|^2$ (this implies adding the probabilities for $a$- and $b$-orbitals on each site). The state localized at the upper (lower) edge corresponds to the negative (positive) slope in (b) and has a negative (positive) group velocity, i.e., leads to transport to the left (right).

Let us now introduce edges to our model to find the edge state predicted by the topological properties of the Hamiltonian (4.1). Instead of assuming translational invariance in both directions of our two-dimensional system, we keep the translational invariance in $x$-direction (and thus $k_x$ as a “good” quantum number), but consider only a finite number of sites in $y$-direction (labeled with the index $n$). This system now features two edges (an upper and a lower edge in $y$-direction, see sketch in Fig. 4.2(b)). Introducing $\hat{a}_{nm} = \frac{1}{\sqrt{V}} \sum_{k_x} e^{i k_x n} \hat{a}_{k_x,n}$

$$
Q = \begin{cases} 
0 & \text{if } 2 < W/B \\
+1 & \text{if } 0 < W/B < 2 \\
-1 & \text{if } -2 < W/B < 0 \\
0 & \text{if } W/B < -2
\end{cases}
$$

i.e., at most one edge state per edge.
4. Classical nonlinear dynamics of topological transport in bosonic models

and \( \hat{b}_{nm} = \frac{1}{\sqrt{V}} \sum_{k_x} e^{ik_x m} \hat{b}_{k_x n} \) into Eq. (4.1) leads to the following Hamiltonian for a one-dimensional strip,

\[
\hat{H}_{1D} = \sum_{k_x, n} \left[ W \left( \hat{a}_n^\dagger \hat{a}_n - \hat{b}^\dagger_n \hat{b}_n \right) + \frac{B}{2} \left( \hat{a}^\dagger_{n+1} \hat{a}_n - \hat{b}^\dagger_{n+1} \hat{b}_n + h.c. \right) \right] + B \cos(k_x) \left( \hat{a}_n^\dagger \hat{a}_n - \hat{b}_n^\dagger \hat{b}_n \right) - A \sin(k_x) \left( \hat{a}_n^\dagger \hat{b}_n + \hat{b}_n^\dagger \hat{a}_n \right) - \frac{A}{2} \left( \hat{a}^\dagger_{n+1} \hat{b}_n - \hat{b}^\dagger_{n+1} \hat{a}_n + h.c. \right),
\]

where we now suppressed the \( k_x \) index of the operators. This Hamiltonian can be diagonalized to obtain its eigenvalues, i.e., the band structure and the corresponding eigenstates. Because of the finite number of sites in the \( y \)-direction the 1D-band structure, see Fig. 4.2(b), consists of separate lines (their number is determined by the number of sites and orbitals per site) that are bunched together into bulk bands (shown in grey). The edge states, marked in red, are the only states that have energies lying inside the former bandgap. The Chern number for the shown parameters is \( Q = -1 \), i.e., there has to be one edge state per edge, which results in two lines crossing the bandgap because of the two edges of the system. In Fig. 4.2(c) we show that the corresponding eigenstates of the system are indeed localized at the respective edge when the corresponding eigenenergy is well within the bandgap (colored in red). Here, this is the case for quasimomenta around \( k_x = 0 \). Furthermore, the sign of the Chern number, i.e., \( Q < 0 \) determines the chirality to be anti-clockwise. Thus, the state localized at the upper edge moves to the left, which is also visible in Fig. 4.2(b), where the slope of the corresponding eigenvalue is negative. Accordingly, the line crossing the bandgap with a positive slope corresponds to a right-moving state, localized at the lower edge.

4.4 Nonlinear model

As seen in the previous section, a (Chern) topological insulator can be realized in a linear model, i.e., corresponds to a bilinear Hamiltonian. We now want to study how such linearly generated topological transport is affected by nonlinear interactions and explore the corresponding nonlinear dynamics. In particular, we investigate the dynamics of a topological Chern insulator in the presence of a local nonlinearity. Our nonlinear model is based on the above explained half BHZ model which features well-known chiral edge states. We consider the classical equations of motion associated to Hamiltonian (4.1), add a cubic nonlinear term, and find
Figure 4.3: Exciting a nonlinear topological insulator. The initial conditions, i.e., the way of exciting states in a nonlinear system determine which aspects of the nonlinear system can be studied. Possible approaches are: (a) To start with a specific state on the lattice, e.g. a linear edge state. (b) To launch a wavepacket at the edge of an empty lattice. The wavepacket could also be positioned in the bulk instead. (c) To continuously drive the system at one (or several) sites, typically also including damping.

\[ i \ddot{a}_{nm} = W a_{nm} + \frac{B}{2} (a_{nm+1} + a_{n+1m} + a_{n-1m} + a_{nm-1}) + \frac{A}{2} (ib_{nm+1} + b_{n+1m} - ib_{nm-1} - b_{n-1m}) + \beta |a_{nm}|^2 a_{nm} \]  

\[ i \ddot{b}_{nm} = -W b_{nm} - \frac{B}{2} (b_{nm+1} + b_{n+1m} + b_{n-1m} + b_{nm-1}) + \frac{A}{2} (ia_{nm+1} - a_{n+1m} - ia_{nm-1} + a_{n-1m}) + \beta |b_{nm}|^2 b_{nm}. \]  

(4.6)

This can also be written in a more concise form:

\[ i \dot{\Psi}_j = \sum_l H_{jl} \Psi_j + \beta |\Psi_j|^2 \Psi_j. \]  

(4.7)

Here, \( \Psi = (a_{11} b_{11} a_{12} b_{12} \ldots a_{NM} b_{NM})^\top \) is a vector containing the classical amplitudes of both orbitals on all \( M \times N \) lattice sites and \( H \) is a matrix containing the mass term and all linear coupling elements described by Hamiltonian (4.1). The last term is a local cubic nonlinearity which can be understood as a Kerr nonlinearity (without cross-Kerr coupling, i.e., no \( |a_{nm}|^2 b_{nm} \) or \( |b_{nm}|^2 a_{nm} \) terms), coming from a Hamiltonian \( \hat{H}_{NL} = \sum_{n,m} a_{nm}^\dagger a_{nm} a_{nm}^\dagger a_{nm} + b_{nm}^\dagger b_{nm}^\dagger b_{nm} b_{nm} \). The strength of the nonlinearity is determined by \( \beta \), however its relevance also depends on the local intensity \( |\Psi_j|^2 \) via \( |\Psi_j|^2 \Psi_j \).

Note that a linear system is fully characterized by investigating the eigenstates and eigenvalues of the corresponding Hamiltonian. If we know the band structure, we can predict the behaviour of this system no matter how it is actually excited. In contrast, this is no longer possible for a nonlinear system where the concept of a usual band structure fails. One can study several physically different situations, see illustrations in Fig. 4.3, to learn about differ-
4. Classical nonlinear dynamics of topological transport in bosonic models

Figure 4.4: System and initial state. (a) Sketch of the lattice configuration used for numerical simulations: A quasi-infinite strip with periodic boundary conditions in $x$-direction and edges in $y$-direction. (b) We use the linear edge state localized at the upper edge as an initial state and distribute weak Gaussian noise on top of it. The grid has dimensions $M \times N = 84 \times 15$. We show the total intensity $|a_{nm}|^2 + |b_{nm}|^2$ on each site, which is (apart from the noise) smoothly distributed along the edge. (c) and (d) show the real and imaginary part of the complex amplitude of the $a$-orbital on the sites close to the upper edge ($b$-orbitals look similar). Here the choice of $k_0 d = -\pi/7$ becomes visible in the spatial oscillation. Other parameters: $W/B = -1.5$, $A/B = 1$, and the standard deviation of the initial Gaussian noise $\sigma = 0.005$.

ent aspects of the nonlinear dynamics. While we will choose to initially populate the lattice with a linear edge state, several other options exist. One could instead launch a wavepacket, either at the edge or also in the bulk. Furthermore, one could periodically drive one (or several) lattice sites - for example at the edge with a frequency in the linear bulk bandgap, thus continuously exciting the linear edge state. Note that Eq. (4.7) describes a closed system, but can easily be expanded to include, e.g., continuous driving and damping.

In the following, we study the nonlinear dynamics of an empty lattice of dimensions $M \times N$ on which we prepare a specific initial state. To avoid, e.g., effects due to the corners of such a rectangular sample, we consider a strip with quasi-infinite extension in $x$-direction and a finite number of sites in $y$-direction, Fig. 4.4(a). This is achieved by simulating a sufficiently large number of sites in $x$-direction and then introducing periodic boundary conditions between the leftmost and rightmost site. The system is initially populated with a linear edge state obtained for an (truly) infinite strip, cf. Sec. 4.3. In particular, we choose the state located at the upper edge of the system and distribute weak Gaussian noise on top of this, see Fig. 4.4(b). This Gaussian noise acts as a seed for the nonlinear behaviour.

Let us add some technical remarks before moving on to our results: As discussed in
4.5 Linear stability analysis on a strip

Sec. 4.3, the ansatz for an infinite strip relies on the translational invariance in one direction. We choose one linear edge state $\Psi$ within the bulk bandgap by fixing a suitable $k_0$ and determining the eigenstate of Hamiltonian (4.5) that is localized at the upper edge. To transfer this as an initial state to our quasi-infinite strip, we have to use our ansatz, i.e., $a_{mn} = e^{i k_0 m} a_{k_0 n}$ and $b_{mn} = e^{i k_0 m} b_{k_0 n}$, to find the amplitudes on the sites along the $x$-direction. The involvement of periodic boundary conditions leads to the condition $k_x = 2\pi m / M$ where $m \in \mathbb{N}$, i.e., only certain values $k_x = k_0$ are allowed on a lattice with $M$ sites in $x$-direction. This ensures a smooth transition from the leftmost site to the rightmost site of the lattice and can be seen in the complex amplitudes which show spatial oscillations along the edge, Figs. 4.4(c) and (d). In contrast, the intensity on each column of the strip is identical apart from the Gaussian noise that we distribute on top of the actual edge state, Fig. 4.4(b).

In the following section 4.5 we want to gain some insight into the expected behaviour of this situation via a linear stability analysis. Then, in Sec. 4.6, we present a numerical simulation of the above described quasi-infinite lattice and observe the effects of the nonlinearity as time evolves.

4.5 Linear stability analysis on a strip

In this section we investigate the stability of the linear edge state on a strip with respect to the nonlinear equations of motion. The idea is to consider effects of a relatively weak nonlinearity, which does not affect the linear band structure too much, but enables additional nonlinear processes. Furthermore, it is initially not clear that a stationary steady state to the nonlinear equations and a nonlinear analogue to the linear edge state exists. We note that in Ref. [214] an unstable nonlinear edge state was identified and Ref. [215] even predicts a stable edge state in the nonlinear regime for a $\mathbb{Z}_2$-topological insulator. However, because of the different underlying models for the respective topological insulators these results cannot be straight-forwardly transferred to our system. Thus, we start with a linear edge state, which is an eigenvector of the matrix $H$, i.e., $H\Psi = \omega_0 \Psi$, and solves the linear equation $i\dot{\Psi} = H\Psi$ trivially via $\Psi(t) = e^{-i \omega_0 t} \Psi$.

To determine the stability of the state $\Psi$ with respect to the nonlinear equation of motion (4.7) we make the following ansatz,

$$\Psi(t) = e^{-i \omega_0 t} (\bar{\Psi} + \delta \Psi),$$

and aim for an equation of the form $\delta \dot{\Psi} = -i M \delta \Psi$ (such that real parts of the eigenvalues of the linear stability matrix $M$ correspond to frequencies and imaginary parts to damping). Here, $\delta \Psi$ denotes general deviations from the solution to the linear model. We insert the ansatz into the nonlinear equation of motion (4.7) and find

$$\omega_0 \Psi_j + \omega_0 \delta \Psi_j + i \delta \Psi_j = \sum_l H_{jl} \Psi_j + H_{jl} \delta \Psi_j + \beta |\Psi_j|^2 \Psi_j + \beta |\Psi_j|^2 \delta \Psi_j^* + 2g |\Psi_j|^2 \delta \Psi_j$$

(4.8)
where we neglected terms combining two or more $\delta \Psi_j$ and multiplied the full equation with $e^{i\omega t}$, i.e., switched to a frame rotating at $\omega_0$. Because $\Psi$ is assumed to be an eigenvector of the linear matrix $H$, we use $\omega_0 \Psi = H \Psi$ and drop these terms on the right and left hand side of Eq. (4.8). Furthermore, we neglect a constant shift due to the nonlinear term $\beta |\Psi_j|^2 \delta \Psi_j$. This leads to

$$i \delta \Psi_j = \sum_l (H_{jl} - \omega_0) \delta \Psi_j + \beta \left( 2 |\Psi_j|^2 \delta \Psi_j + \Psi_j^2 \delta \Psi_j^* \right).$$

(4.9)

To proceed we make use of the translational symmetry in $x$-direction, cf. Sec. 4.4. Then, the vector $\Psi$ contains all amplitudes $\Psi = (a_1 b_1 a_2 b_2 \ldots a_N b_N)^\top$ on a single column of the lattice, such that the index now labels the sites in $y$-direction only. The full solution $\Psi_j(x)$ can be expressed as $\Psi_j(x) = e^{ik_0x} \Psi_j$ where $x = n \cdot d$ is the position in the infinitely extended direction and $d$ denotes the lattice constant (which we set to 1). While the analyzed state $\Psi$ corresponds to a specific quasimomentum $k_0$, its stability has to be investigated with respect to deviations of all possible quasimomenta $q$. Thus, we make the following ansatz,

$$\delta \Psi = e^{ik_0x} \left( e^{iqx} \delta \Psi_+ + e^{-iqx} \delta \Psi_- \right),$$

(4.10)

and insert it into (4.9). Splitting the result into one equation for terms rotating with $e^{i(k_0+q)x}$ and another equation for terms rotating with $e^{i(k_0-q)x}$ we find

$$i \delta \Psi_+ = \sum_l (H_{jl}(k_0+q) - \omega_0) \delta \Psi_+ + 2 \beta |\Psi_j|^2 \delta \Psi_+ + \beta \Psi_j^2 \delta \Psi_j^*$$

$$i \delta \Psi_- = \sum_l (H_{jl}(k_0-q) - \omega_0) \delta \Psi_- + 2 \beta |\Psi_j|^2 \delta \Psi_- + \beta \Psi_j^2 \delta \Psi_j^*.$$  

(4.11)

Note that $H(k_0)$ is a matrix containing all linear terms originating from the Hamiltonian for a strip, Eq. (4.5), and thus depends on a quasimomentum $k_x$, which in the equations above, is $k_0 \pm q$. Defining the vector $\delta \Psi = (\delta \Psi_+ \delta \Psi_- \delta \Psi^*_+ \delta \Psi^*_-)^\top$ we can finally set up the linear stability matrix $M$ and analyze its eigenvalues $\lambda$ for different $k_0$ and $q$ values.

$$\begin{pmatrix}
    \delta \Psi_+ \\
    \delta \Psi_- \\
    \delta \Psi^*_+ \\
    \delta \Psi^*_-
\end{pmatrix} = -i
\begin{pmatrix}
    H(k_0+q) & -\omega_0 + 2 \beta |\Psi_j|^2 & \beta \Psi_j^2 & 0 \\
    -\omega_0 + 2 \beta |\Psi_j|^2 & H(k_0-q) & 0 & \beta \Psi_j^2 \\
    \beta \Psi_j^2 & 0 & -H(k_0+q) & -\omega_0 + 2 \beta |\Psi_j|^2 \\
    0 & \beta \Psi_j^2 & -\omega_0 + 2 \beta |\Psi_j|^2 & -H(k_0-q)
\end{pmatrix}\begin{pmatrix}
    \delta \Psi_+ \\
    \delta \Psi_- \\
    \delta \Psi^*_+ \\
    \delta \Psi^*_-
\end{pmatrix}$$

(4.12)

Here, the index $j$ inside the matrix, $|\Psi_j|^2$ or $\Psi_j^2$, indicates that the (absolute) square of the entries of the vector $\Psi$ is meant, cf. Eq. (4.11). This linear stability matrix features $2 \times N \times 4$
4.5. Linear stability analysis on a strip

Figure 4.5: Instabilities due to nonlinearity. (a) and (c) show nonlinearity-enabled, unstable scattering processes from the linear edge state marked with a blue cross in the linear band structure (grey). The color code shows the imaginary part of the corresponding eigenvalues of the linear stability matrix, i.e., quantifies the instability. Only unstable eigenvalues, i.e. where \( \text{Im}[\lambda/B] > 0 \), are shown. Unstable energies/quasimomenta appear in pairs with \( \pm q \) and \( \pm \omega \) (as indicated for one example). (b) and (d) show unstable energies of the same linear edge state as in (a) \( k_0 d = -\pi/7 \) and (c) \( k_0 d = \pi/7 \), as a function of the nonlinearity strength \( \beta/B \). Parameters: \( W/B = -1.5, A/B = 1 \), and for (a) and (c) \( \beta/B = 0.05 \).
eigenvalues for each \( k_0 \) and \( q \), where \( 2 \times N \) is the dimensionality of the strip of \( N \) sites with two orbitals each and the factor \( \times 4 \) is due to the four deviations \( \delta \Psi_+ \), \( \delta \Psi_- \), \( \delta \Psi^*_+ \), \( \delta \Psi^*_- \).

We can now analyze the stability of any edge state for a fixed \( k_0 \) by determining the eigenvalues of the linear stability matrix \( M \) for all quasimomenta \( q \) within the first Brillouin zone. The bare eigenvalues of the above defined matrix \( M \) have to be understood with respect to the analyzed state and thus have to be shifted appropriately in quasimomentum and energy to be displayed in Fig. 4.5. In Figs. 4.5(a) and (c) we show (in grey) the linear band structure. The blue cross indicates \( k_0 \) and the corresponding energy of the edge state \( \Psi \) for which we perform the linear stability analysis. If the nonlinearity is weak we expect the linear band structure to be a good approximation of available energy levels. The nonlinearity, due to its quartic nature on the Hamiltonian level, allows for scattering processes creating two excitations at once. To fulfill momentum and energy conservation we can expect processes that create excitations at \( \pm q \) and \( \pm \omega \) at the same time and end up at (or at least close to) existing bands. All other processes are highly suppressed. Additionally, some processes (that would fulfill the just stated criteria) are suppressed because the mode overlap between the analyzed edge state and the target state is small. As an extreme example consider a scattering process to the edge state at the other (in this scenario lower) edge, which might be allowed by energy and momentum conservation. Still, the spatial separation of these localized modes highly suppresses such scattering processes.

In Fig. 4.5(a) and (c) we find exactly this behaviour: Instabilities show up symmetrically around the analyzed state (preserving energy and momentum), but only close to existing bands. Note that the shown instabilities are actually grouped in small regions around favourable transitions. These separate small regions come about due to the finite number of sites in \( y \)-direction and thus the separation of energy states in the band structure.

The interpretation of instabilities as combined scattering processes towards states at \( \pm q \) and \( \pm \omega \) with respect to \( k_0 \) is also supported by the eigenstates corresponding to the unstable eigenvalues. Most eigenvectors \( \delta \tilde{\Psi} \) represent coupling to one sort of fluctuations only, i.e., they have (significantly) non-vanishing entries only at \( \delta \Psi_+ \) or \( \delta \Psi_- \) or \( \delta \Psi^*_+ \) or \( \delta \Psi^*_- \). The associated eigenvalues of such eigenvectors have vanishing imaginary parts and are thus considered stable. In contrast, non-vanishing imaginary parts come along with eigenvectors describing (almost equal) coupling to two types of fluctuations: Notably, only the combinations \( \delta \Psi_+ \) with \( \delta \Psi^*_+ \), and \( \delta \Psi_- \) with \( \delta \Psi^*_- \) appear. This is again in agreement with momentum and energy conservation since \( \pm \) label the opposite shifts in quasimomentum and the normal and complex conjugate \( \delta \Psi \) correspond to opposite shifts in energy.

It is rather surprising to find that the state \( \Psi \) corresponding to \( k_0 \) is stable with respect to quasimomenta that deviate only marginally from \( k_0 \), i.e., with very small \( q \), which naively should fulfill the above stated criteria of momentum and energy conservation. In both Figs. 4.5(a) and (c), which show the instabilities for edge states of different \( k_0 \), i.e. \( k_0 \equiv -\pi/7 \) in (a) and \( k_0 \equiv +\pi/7 \) in (c), there is a finite gap between \( k_0 \) and the first unstable quasimomentum \( k_0 \pm q_{\text{min}} \). This gap can also be seen in Figs. 4.5(b) and (d) for different strengths of the nonlinearity \( \beta \) (corresponding to the analysis of the same edge state as in (a) and (c).
4.6. Nonlinear time evolution

respectively). Note, that there the unstable energies (not quasimomenta) are shown as function of $\beta$. The colorscale again quantifies the instability of the energies, i.e., corresponds to the imaginary part of the eigenvalues. Only unstable energies (Im[$\lambda/B$] > 0) are shown. We attribute the small, disconnected values appearing, e.g., at small $\beta$ and large energies or the single pair of points sitting close to the energy of the analyzed edge state (blue line) to numerical inaccuracies. The overall behaviour both in Figs. 4.5(b) and (d) shows that the instabilities increase with $\beta$ and the unstable regions broaden. This broadening occurs since a stronger nonlinearity enables also more processes in the vicinity of transitions exactly matching the conditions. Moreover, the unstable energies, especially the ones closest to the energy of the analyzed edge state $\bar{\Psi}$, are shifted with increasing $\beta$. Interestingly, we notice a distinct difference between Fig. 4.5(b), where we show the behaviour corresponding to the edge state with $k_0d = -\pi/7$, and Fig. 4.5(d), where we show the behaviour corresponding to the edge state with $k_0d = +\pi/7$. In the former, the energies are shifted further away from the energy of the edge state, while in the latter they are drawn closer. We generally observe this asymmetry in behaviour for $k_0d > 0$ and $k_0d < 0$, but notably the roles are interchanged if we investigate the state localized at the lower edge instead of the upper edge state. This implies that the behaviour of the upper and lower edge state is the same when their energies are the same, i.e., their quasimomenta have opposite sign. It is also interesting to note that the behaviour of $k_0d > 0$ and $k_0d < 0$ at the same edge is interchanged if the sign of the nonlinearity $\beta$ is flipped (changing from a repulsive to an attractive nonlinearity).

The results of this analysis do not only depend on the strength of the nonlinearity $\beta$ but also on the amplitudes of the analyzed state $\bar{\Psi}$ on all sites. Thus, if the processes described here lead to a redistribution of amplitude, then this analysis should be repeated for the modified state. In particular, if we now move on to the explicit time evolution of a strip with finite extension also in $x$-direction (and periodic boundary conditions), then the nonlinear interaction leads to redistributions and to significant accumulation of intensity at certain sites of the edge. Eventually such processes drive the system towards a nonlinear regime that is no longer captured within this linearized description at all. Nonetheless, the unstable processes identified in this section allow us to predict the amplification and thus appearance of certain wavelengths on top of the initial edge state on an intermediate timescale. We discuss this in the next section.

4.6 Nonlinear time evolution

Let us now move on to the full numerical simulation of a quasi-infinite strip, as described in Sec. 4.4 and shown in Fig. 4.4. We want to gain an overview about the dynamical behaviour of the nonlinear system and in particular confirm our results from the linear stability analysis. Therefore, we start off by populating the lattice with a linear edge state at the upper edge and initially add weak random Gaussian noise at all sites. We simulate the dynamics according to Eq. (4.7), see Appendix A.3 for details on the numerical method.
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In Fig. 4.6 we show an overview of the time evolution. Since the linear edge state is unstable with respect to the nonlinear dynamics, see Sec. 4.5, the bulk modes become significantly occupied in the long-time limit, Fig. 4.6(a). In particular, we show how the bulk and edge occupancy change as time evolves. Here we include the two uppermost lattice sites into the definition of “edge” occupancy, i.e., we calculate \( \sum_m (|a_{1m}|^2 + |a_{2m}|^2 + |b_{1m}|^2 + |b_{2m}|^2) \). The bulk occupancy is simply taken to be the rest of the lattice since the lower edge state is not occupied at all. Notably, Fig. 4.6(a), the bulk and edge occupancies remain unchanged for some intermediate time. Meanwhile the actual distribution along the edge, shown in Figs. 4.6(b)-(e) for short time periods, is modified due to nonlinear processes. While in Fig. 4.6(b) the initial random noise distribution on top of the edge state still dominates, soon a specific modulation of the edge state appears, see Fig. 4.6(b) and even clearer in Fig. 4.6(d). This modulation can be understood with the help of the processes determined within the linear stability in Sec. 4.5, and will be discussed below. As the amplitude of the oscillation around
the edge state grows, the nonlinearity becomes stronger and stronger at the highly occupied sites, finally giving rise to a less regular pattern, Fig. 4.6(e). It can no longer be viewed as a modulation of the initial linear edge state as it is a mixture of highly occupied and almost empty sites. At this stage of the time evolution, significant loss of edge occupation into bulk modes occurs, Fig. 4.6(a). Eventually, the bulk modes become more occupied than the original linear edge state.

Let us now analyze the spatial oscillation appearing along the edge before the nonlinearity and the instability towards the bulk become too strong. With the ansatz $\Psi(t) = e^{-i\omega t} (\Psi + \delta\Psi)$ from the linear stability analysis, where $\delta\Psi = e^{i k_0 x} (e^{i q x} \delta\Psi_+ + e^{-i q x} \delta\Psi_-)$, the observed intensity $|\Psi_j|^2 \propto \cos (2 q x)$ is expected to show modulations with a wavelength $2\pi / 2q d$, i.e., a wavelength associated to $2q$ where $q$ denotes the quasimomentum of the amplified fluctuation.

In Fig. 4.7(a)-(c) we show again the $a$-orbital occupation along the upper edge at three points in time ($b$-orbitals look similar). It becomes clearly visible how the initial noise ripples on top of the edge state turn into a periodic modulation. In principle, many wavelengths are
involves in this process. Already from the linear stability analysis it is evident that the edge state is unstable with respect to several quasimomenta and thus the associated wavelengths. However, the imaginary parts of the eigenvalues in the linear stability analysis determine how strongly a specific process is amplified. Thus, one can expect that the wavelength (quasimomentum $q$) associated to the largest imaginary part eventually shows up. In Fig. 4.7(c), shortly before the edge state starts to become significantly unstable with respect to the bulk, a strongly amplified wavelength can be readily determined. In Fig. 4.7(d) we show the imaginary parts of unstable quasimomenta and indicate the quasimomentum of fluctuations that leads to the observed spatial wavelength as dashed lines. It is indeed close to the $q$s from the linear stability analysis that are highly unstable.

Note that the observed wavelength is not completely independent of the chosen lattice size. In particular, for the parameters presented here, the highly unstable processes are associated to small quasimomenta of the fluctuations and thus rather large wavelengths. On smaller grids, these wavelengths might not show up simply because they scarcely fit on the simulated lattice or match badly with the periodic boundary conditions enforcing a smooth transition from the leftmost to the rightmost site. Also the height of the strip, i.e., the number of sites in $y$-direction, has some influence on the overall behaviour. A larger height leads to additional unstable quasimomenta, but also affects the behaviour of radiation into the bulk.

### 4.7 Concluding remarks

In conclusion, we have analyzed the dynamical behaviour of a bosonic nonlinear topological insulator. Based on the half BHZ model with a local Kerr nonlinearity we investigated the scattering processes induced by a rather weak nonlinearity, giving rise to specific spatial modulations of an initial (linear) edge state. With a linear stability analysis we can identify the wavelengths that are amplified before the amplitude of the modulation grows so much that strong nonlinear effects occur and destroy the spatial periodicity. In this regime, the occupation at the edge, which is mostly preserved during the initial phase of nonlinear modulation, starts to significantly radiate into the bulk. At sufficiently long times, the bulk modes become more occupied than the original edge state.

Concerning our results, we still aim for a better understanding of the unstable scattering processes. Here, the maximal instability, the wavelength or quasimomentum $k_0 \pm q$ associated to this maximal instability, and the onset of instability, i.e., the smallest unstable $|q|$, are of special interest. We have discussed the dependence on the nonlinearity $\beta$ and the role of an attractive as compared to a repulsive nonlinearity, which requires some further investigation. Furthermore, the results depend on the analyzed edge state $\Psi$ and thus on the quasimomentum $k_0$ associated to this state. In first simulations (not shown) we see a strong dependence of the maximal instability, the associated wavelength, and the onset of instability on $k_0$, however further investigation is needed. Finally, there are also the BHZ model parameters, i.e., $A/B$ or $W/B$. In particular, $W/B$ determines the Chern number and thus whether edge states
4.7. Concluding remarks

even exist. Generally speaking, both $A/B$ and $W/B$ modify the band structure (even if edge states exist) and thus at least influence quantitatively the scattering processes.

In comparison to literature, in particular the finite onset of instability that we observe for large parameter ranges is surprising. In Ref. [214] a region of instability starting immediately for $|q| \neq 0$ is shown. However, in contrast to our work, this analysis is based on a continuous model for a honeycomb array and the analyzed state is an edge state of the nonlinear model itself. It would be useful to identify what exactly is causing our different findings. Furthermore, since solitons (mostly edge solitons) have been discussed a lot in literature so far, it would be worthwhile to try to identify solitonic behaviour in the strongly nonlinear regime of our model. It would be interesting to know whether this is already contained in this simple nonlinear half BHZ model that we investigated. Therefore, one could also study the evolution of a wavepacket launched at the edge of the system, instead of starting from a linear edge state. Finally, in combination with a careful analysis of potential finite-size effects, it would also be interesting to understand the phase of radiation into the bulk that displays several “plateaus” in the decay (growth) of the edge (bulk) occupancy in our simulation.
4. Classical nonlinear dynamics of topological transport in bosonic models
In the first part of this thesis we have discussed several aspects of quantum synchronization. First, in Ch. 2, we have investigated the effects of quantum (and classical) noise on the mutual synchronization of two optomechanical systems. We have shown that different synchronization regimes (in-phase, anti-phase, and bistable synchronization) can still be identified in the quantum regime, although quantum noise prevents exact phase locking. Moreover, we have found noise-induced transitions between two synchronization states and explored the corresponding classical-to-quantum crossover. Notably, we have shown, that such transitions can also be observed in situations where in the classical, noiseless limit only one stable synchronization state exists, i.e., we have found noise-induced bistable behaviour.

Then, in Ch. 3, we have turned to the quantum Van der Pol oscillator as a paradigmatic system for limit-cycle oscillators in order to identify genuine quantum features. In particular, we have studied the synchronization dynamics of a quantum Van der Pol oscillator coupled to an external reference and derived an effective model. This has allowed us to distinguish different dynamical regimes and, most notably, we have found a regime of quantum-coherent phase motion. In this regime, quantum signatures like negative Wigner densities, are preserved for many oscillations of the system. As an additional result, we have also found self-sustained oscillations of the phase in the quantum regime instead of the (slowly) damped phase dynamics of synchronization. This can still be viewed as a synchronized state in an averaged sense, since the phase dynamics is amplitude-bounded to a subinterval between 0 and 2\pi. It would be interesting to further explore the properties of these phase self-oscillations. However, most importantly, our finding of quantum-coherent synchronization dynamics opens up new prospects to study quantum phenomena in the context of synchronization. Previous works focused on the behaviour of synchronization in quantum sys-
systems and the observed modifications of synchronization were not necessarily non-classical themselves. Quantum-coherent synchronization dynamics now allows to explore and exploit quantum states throughout the dynamics itself. For example, one might envision that there are not only transitions between synchronization states which are understood as the noise pushing the system over a potential barrier, but also quantum tunneling might occur. Furthermore, quantum superpositions between different synchronization states might be achievable. These, and other quantum phenomena, are now accessible for further investigations. Due to the general character of our scheme, we expect that these results can also be transferred to synchronization of two (or more) oscillators and, of course, to several physical systems. Nonetheless, it would be interesting to specifically show and explore this. For example, it would be worthwhile to identify the quantum-coherent regime in optomechanical synchronization. Therefore, but also generally, it would be useful to find a mapping between the effective parameters in the Van der Pol oscillator model and the microscopic optomechanical parameters. Moreover, the rich attractor diagram of optomechanical systems gives rise to additional interesting dynamical regimes. For example, the effects of synchronization in a regime where several limit cycles of, e.g., different amplitude exist could be studied. Of course, it would also be of great interest to be able to expand our models of quantum synchronization to larger (but finite) lattices. This would, e.g., open up possibilities to study quantum phase pattern formation (which is related to synchronization) in a similar sense how it can be done in the classical regime. However, this is computationally extremely challenging and would require smart approaches to make it feasible.

Generally speaking, the field of quantum synchronization would still benefit from the definition of a unique, meaningful measure and a deeper understanding of the interplay between correlations and synchronization. Furthermore, we think that the first experimental demonstration of synchronization in the quantum regime is one of the next important milestones in this field. Recent experimental achievements with nano- and optomechanical systems are promising and show that there is interest in synchronization in this community. Moreover, we also believe that an implementation with trapped ions or atoms should be within experimental reach.

In the remainder of the thesis, Ch. 4, we have then investigated the effects of nonlinearities on the robustness of edge states of Chern topological insulators in bosonic systems. To this end, we have studied the classical nonlinear dynamics of the half Bernevig-Hughes-Zhang model on a square lattice, showing how a linear edge state looses stability in the presence of a local Kerr nonlinearity. In particular, we have presented a linear stability analysis to identify the unstable scattering processes that are enabled by the nonlinear interaction and give rise to spatial oscillations of the edge state. We have observed these spatial oscillations by performing a full time evolution, showing that they are consistent with the results of the linear stability analysis. Furthermore, the full time evolution also reveals that those spatial modulations eventually lead to strong nonlinear effects which destroy the periodicity and give rise to decay into bulk modes. We still aim for a better understanding of the observed instabilities and several parameter dependencies remain to be explored in more detail, e.g., the possibly
different roles of attractive and repulsive nonlinearities. It would also be interesting to find out how much the results are affected if nonlinear cross-Kerr terms are included as well. In order to relate to previous studies, it would be useful to investigate the possible formation of solitons in our model. In general, the interest in effects of nonlinearities on topological insulators in bosonic systems has emerged only very recently. Various topological insulator models with, for example, different lattice geometries have been studied in this context. The central themes are how different topological properties influence nonlinear behaviour (like, e.g., the formation and dynamics of solitons) and, vice versa, how nonlinear phenomena (like solitons) can change (local) topological properties. In the future, it is important to achieve a fundamental, geometry/model-independent understanding of the interplay between nonlinear behaviour and topological properties. Nonetheless, it would also be worthwhile to transfer the investigations from the abstract bosonic lattice models to the specific realization of a nonlinear topological insulator, e.g. by optomechanical means.
Especially the first part of this thesis, Chapter 2, heavily relies on numerical simulations. There are fundamental reasons for this: First, the progress in the field of optomechanics naturally drives theoreticians from analytically accessible problems to more complex questions, e.g., from single optomechanical systems to coupled systems or even large arrays. Second, even if one succeeds in finding an approximate analytical description, often a numerical simulation is applied to check the validity of the simplified approach. In summary, numerical simulations can give us an intuition about the behaviour of a complex system, even in regimes where analytical approximations break down or fail from the beginning.

Having mentioned the merits of numerical simulations, we want to also point out the downsides: Apart from feeding our intuition and allowing for “fits”, similar to how experimentalists treat their data, numerical simulations do not give us easy access to the underlying physics. We can “observe” phenomena in the full model, and yet, not be able to trace these back to their exact physical origin. Parameter-dependences can be “explored” but drawing clear conclusions might be challenging, especially for large parameter spaces. Furthermore, there are also technical limitations to numerical simulations. Finite computational power and memory set restrictions to what is numerically accessible nowadays. Consequently, optimizing numerical methods can be viewed as a research field on its own. One possibility is to carefully choose which representation of a physical system is best suited for computer simulations.

In this appendix, we thus want to summarize the specific numerical methods that were applied throughout this thesis and mention the more technical details of their implementation. We also want to discuss some of their respective advantages and disadvantages that finally resulted in our choice of methods. Note that we do not aim for a general introduc-
A. Numerical methods

A.1 Quantum jump trajectories

Open quantum systems, i.e., quantum systems that interact with an environment, can be treated with several approaches. The Lindblad master equation is maybe the most common method whenever it is applicable. It can either give access to the time evolution of the quantum state or, for equations with time-independent coefficients, be solved to determine directly the steady state. The associated computational challenge heavily depends on the size of the involved matrices representing the physical Hilbert space. Already for a usual harmonic oscillator the Hilbert space is in principle infinite. To numerically tackle the problem one has to truncate the Hilbert space to a reasonable finite size, without creating unphysical results, i.e., one drops states higher than one where the occupation probability distribution has become sufficiently small. In Chapter 2 we have investigated two coupled optomechanical systems, where each optomechanical system consists of two harmonic oscillators. In order to study (small) mechanical limit cycles we chose the Fock basis, needing at least a few photons (this is sufficient because we used very strong optomechanical interaction) and some more phonons. As an example, let us consider five optical states, \(d_{\text{opt}} = 5\), and 30 mechanical states, \(d_{\text{mech}} = 30\). This means that the highest considered photon state is \(|4\rangle\) (since we start with the vacuum) and the occupation probability of this state \(|\langle \Psi | 4 \rangle|^2\) should already be negligible. Similar restrictions apply to the mechanics. Even though this dimensions for a single optomechanical system were chosen to be as small as possible, they already lead to a total Hilbert space dimension \(d_H = d_{\text{opt}}^2 d_{\text{mech}}^2 = 22500\) for the coupled system of two identical optomechanical systems. Note that this is the number of coefficients of the wave function \(|\Psi\rangle\), while the Hamiltonian of the system as well as the density matrix have \(d_H^2 = 506250000\) entries. The action of the master equation on the density matrix requires storing even more coefficients, \(d_H^4\) in fact. Although one can make use of sparse matrix techniques it is clear that this system is not approachable using the Lindblad master equation.

In order to avoid the large memory consumption that comes with the use of the master equation, we have opted to use another standard technique for open quantum systems, so-called \textit{quantum jump trajectories} [178, 179]. This “unraveling” of the Lindblad master equation allows to work with the Schrödinger equation, thus only requiring wave functions of dimension \(d_H\) and a Hamiltonian matrix of dimension \(d_H^2\). Quantum jump trajectories were briefly explained in Sec. 2.4 using the example of a single cavity and the algorithm for this minimalistic example is depicted in Fig. A.1. Dissipation to the environment, in this example photon leakage out of the cavity, is incorporated via a single-photon detector with
100% detection efficiency.\footnote{More sophisticated detection schemes and finite temperature are also realizable with quantum jump trajectories. However, as they were not applied in this thesis, we will not discuss them here.} We will now revisit this basic example with a stronger focus on the actual implementation. At every timestep $\delta t$ of the simulation, a random number is drawn and defines whether a photon leaving the cavity is detected or whether there occurred no loss. To this end, the probability of loosing a photon to the environment is computed at each timestep as $p(t + \delta t) = \kappa \delta t \langle \hat{a}^\dagger \hat{a} \rangle (t)$. Naturally, it depends on the linewidth of the cavity $\kappa$, as well as on the current photon number inside the cavity. It is scaled with the
size of the timesteps $\delta t$. Clearly, this probability has to be sufficiently small to justify that multi-photon losses are neglected. This sets an upper limit on the $\delta t$ that can be chosen for a specific simulation. Now, there are two possible outcomes of the detection event, represented by the random number outcome. (i) If a photon leaving the cavity was detected, the state of the system is updated with this knowledge $|\Psi(t + \delta t)\rangle = \hat{a}|\Psi(t)\rangle$ and normalized afterwards. (ii) If no photon was detected, the state of the system is propagated according to the Schrödinger equation $i\hbar \partial_t |\Psi\rangle = \hat{H}_{\text{eff}} |\Psi\rangle$. However, this propagation is done with an effective non-Hermitian Hamiltonian instead of the bare system Hamiltonian. The solution to the Schrödinger equation is approximately obtained using, e.g., the Euler method, $|\Psi(t + \delta t)\rangle = e^{-i\hat{H}_{\text{eff}}\delta t/\hbar} |\Psi(t)\rangle \approx (1 - i\hat{H}_{\text{eff}}\delta t/\hbar)|\Psi(t)\rangle$. It again requires $\delta t$ to be sufficiently small. In this example $\hat{H}_{\text{eff}} = \hat{H}_0 - i\hbar \frac{\kappa}{2} \hat{a}^\dagger \hat{a}$, with the bare Hamiltonian just a simple harmonic oscillator $\hat{H}_0 = \hbar \omega \hat{a}^\dagger \hat{a}$. The reason for this effective Hamiltonian is the information gain that comes with the absence of a detection event. As a simple example consider a superposition state like $|\Psi\rangle = 1/\sqrt{2} (|1\rangle + |10\rangle)$. Basically, if no photon is lost we learn that (probably) there were not so many photons in the cavity in the first place, i.e., the probability of having many photons in a cavity drops and thus it becomes more likely to find the system in the state $|1\rangle$.

Note that since the effective Hamiltonian is non-Hermitian, the new state has to be normalized before proceeding with the next timestep. The result is a trajectory that would correspond to a single experimental run (with an ideal detector), that is, to one specific temporal trace of the many allowed by quantum mechanics. The predictions of the master equation are recovered by averaging over sufficiently many trajectories and in the limit of infinitely many trajectories the outcomes are indeed equivalent [178]. If the investigated system settles into a steady state it is also possible to obtain steady-state expectation values by averaging a single trajectory over long enough times (not including initial transient behaviour speeds up convergence). Technically speaking, we have now exchanged the large memory consumption needed to compute the Lindblad master equation for the requirement to average over many realizations in the case of quantum jump trajectories. Obviously it is possible to compute several quantum jump trajectories in parallel, however the long computation times remained the main challenge for our system.

Now we want to give some more technical details on the algorithm that we used for the quantum jump simulations in Chapter 2. For the simulations we used a pre-compiled C code via a yorick interface. The considered system consists of two coupled optomechanical systems, thus four separate dissipation processes have to be included: photon loss from cavity 1, photon loss from cavity 2, phonon loss of the mechanical oscillator 1, and phonon loss of the mechanical oscillator 2. Each loss probability is calculated in analogy to the example of a single cavity. However, now we have to ensure that the combined probabilities are sufficiently small, i.e., also the joint losses of a photon and phonon etc. are not included in this method and thus the corresponding probabilities need to be sufficiently small. Note that the detectors (represented by the random number) also distinguish between photons leaving cavity 1 and cavity 2 and accordingly for the mechanics. The respective loss is then realized by applying the corresponding jump operator ($\hat{a}_1, \hat{a}_2, \hat{b}_1, \text{or} \hat{b}_2$) to the current state
A.2. Stochastic Langevin equations and “mimicked” quantum noise

\[ |\Psi\rangle \] The effective Hamiltonian is obtained by adding four terms to the bare Hamiltonian Eq. (2.4), i.e. \( \hat{H}_{\text{eff}} = \hat{H}_0 - i\hbar \frac{\kappa}{2} \hat{a}_1 \hat{a}_1^\dagger - i\hbar \frac{\kappa_2}{2} \hat{b}_1 \hat{b}_1^\dagger - i\hbar \frac{\Gamma}{2} \hat{b}_2 \hat{b}_2^\dagger \). To solve the Schrödinger equation we switched from using the Euler method to the fourth-order Runge-Kutta scheme in order to use slightly larger timesteps \( \delta t \). The Hilbert space is described in the number basis, i.e. with states \( |n_1, n_2, m_1, m_2\rangle \), where \( n_1, n_2 \) label the photon number states in cavity 1 and 2 and \( m_1, m_2 \) label the phonon number states of the mechanical oscillators 1 and 2, respectively. Maximal values for optical and mechanical excitations were chosen depending on the specific simulation. A hashing procedure was used to map the four quantum numbers \( n_1, n_2, m_1, m_2 \) to a unique integer identifier for the position in the state vector. In particular the final index \( Q_{\text{tot}} = Q_{\text{opt}} d_{\text{mech}}^2 + Q_{\text{mech}} \) consists of an optical and mechanical index and depends on the maximum number of mechanical states that were included for each of the mechanical oscillators. Furthermore, \( Q_{\text{opt}} = n_1 d_{\text{opt}} + n_2 \) and \( Q_{\text{mech}} = m_1 d_{\text{mech}} + m_2 \), each depend on the actual quantum numbers and the maximum number of optical or mechanical states. The Hamiltonian and jump operators were stored as sparse matrices using the compressed row storage method. The intention is not only to save memory, but mainly to minimize the number of multiplications when applying an operator to the state. The basic idea of compressed row storage is to replace a sparse matrix by three vectors, where one consists of all non-zero values of the matrix, one stores the column-index of each non-zero value, and one stores the index of the value-vector where a new row starts.

In summary, using quantum jump trajectories enabled us to study the full nonlinear problem of two coupled optomechanical systems. However, careful programming and a narrow choice of physical and computational parameters was necessary to keep the computational time within a realistic timeframe.

A.2 Stochastic Langevin equations and “mimicked” quantum noise

In the last section we explained the advantages and difficulties of quantum jump trajectories. Although the simulation of the full nonlinear quantum system became feasible in certain regimes, using quantum jump trajectories was neither very convenient nor did it allow us to sweep a large parameter space. Actually most of the results presented in Ch. 2 were achieved simulating the semi-classical equations (2.5) instead. Quantum jump simulations were then used to confirm our trust in the semi-classical approach, at least from a qualitative point of view. The equations (2.5) are stochastic Langevin equations. They contain an additive noise term that has a Gaussian probability distribution. As explained in Ch. 2.4, these classical Gaussian white noise terms are designed to “mimic” quantum noise. This means that the quantum noise correlators

\[ \langle \hat{a}_1^\dagger (t) \hat{a}_1 (t') \rangle = 0 \quad \text{and} \quad \langle \hat{a}_1 (t) \hat{a}_1^\dagger (t') \rangle = \delta (t - t') \]  

are replaced by classical noise correlators

\[ \langle \alpha_1^\dagger (t) \alpha_1 (t') \rangle = \langle \alpha_1^\dagger (t) \alpha_1 (t') \rangle = \delta \{ t - t' \} / 2. \]  

(A.2)
A. Numerical methods

They have to deviate from “real” quantum noise correlations because complex numbers cannot fulfill the commutation relations of quantum operators and thus the two correlators given in Eq. (A.2) cannot be distinguished. We will discuss the consequences of this below. First, let us turn to the numerical implementation: If a noise term is added to a deterministic differential equation, it is implemented by drawing random numbers from an appropriate distribution. To represent the delta-correlated Gaussian white noise (A.2), at each timestep of the numerical integration a new random number has to be drawn from a normal distribution of zero mean and a variance depending on the size of the timestep $\delta t$. To be more precise, here we need to rescale the variance with $1/2\delta t$, which corresponds to multiplying the actual random number with $1/\sqrt{2\delta t}$. Additionally, the actual integration method, e.g. the Euler method discussed in the previous section, brings in an additional $\delta t$ dependence, such that the overall scaling of the random number term in the equation is proportional to $\sqrt{\delta t}$. However, using a standard Euler method to solve a stochastic differential equation is not very efficient. Generally, when it comes to stochastic differential equations, standard numerical integration methods have to be applied very carefully and are much less efficient. The Gaussian white noise term is indeed (infinitely) fast fluctuating with time, such that a standard Euler integration would require extremely tiny timesteps to achieve reasonable precision. This, however, corresponds to a significant increase of the required computation time. Instead, we turn to numerical integration methods specifically developed for solving stochastic differential equations, see e.g. Ref. [219] for an overview. Throughout the work on this thesis we used two different stochastic “Runge-Kutta”-schemes, first one of order 1 [220] and later one of the order 1.5 [221]. We have chosen these particular schemes because they were easy to implement and turned out to converge sufficiently fast for our purpose.

Both schemes rely on several random numbers that are drawn at each timestep but speed up convergence as compared to non-specialized methods. Normally distributed random numbers were obtained with yorick’s built-in function or, in C, with an implementation of the Box-Müller algorithm relying on a built-in function for equally spaced random numbers. Note that Eqs. (2.5) are complex and, therefore, the noise term must be complex as well. The real and imaginary parts of the noise are independent and can thus be drawn as separate random numbers. However, the variance of the respective probability distributions has to be adapted. A simple calculation, $\langle \text{Re}[\alpha_{\text{in}}(t)] \text{Re}[\alpha_{\text{in}}(t')] \rangle = \frac{1}{2} \langle \alpha_{\text{in}}^\dagger(t) \alpha_{\text{in}}(t') \rangle + \frac{1}{2} \langle \alpha_{\text{in}}(t) \alpha_{\text{in}}^\dagger(t') \rangle$ and similar for the imaginary part, shows that the variances of the real and imaginary part are the variance of the full (complex) noise simply rescaled with a factor 1/2. It is crucial to collect all rescaling factors correctly and apply them either to the random number drawn from a standard normal distribution of variance 1, or to directly draw random numbers from a distribution of rescaled variance.

At this point, we want to highlight a peculiarity that comes along with the mimicked noise terms. Let us consider an “empty” cavity as an illustrative example, where “empty” means not driven and not coupled to any other system. It only couples to the vacuum environment, thus

\[ \langle \hat{a}_{\text{in}} \rangle = \langle \hat{a}_{\text{in}}^\dagger \rangle = 0, \text{ and similarly } \langle \alpha_{\text{in}} \rangle = \langle \alpha_{\text{in}}^\dagger \rangle = 0. \]
A.2. Stochastic Langevin equations and “mimicked” quantum noise

brining in quantum fluctuations. The corresponding quantum Langevin equation is

\[ \dot{a} = -i \left( \omega_c + \frac{\kappa}{2} \right) a - \sqrt{\kappa} a_{\text{in}}, \tag{A.3} \]

while the corresponding semi-classical Langevin equation would be

\[ \dot{\alpha} = -i \left( \omega_c + \frac{\kappa}{2} \right) \alpha - \sqrt{\kappa} \alpha_{\text{in}}. \tag{A.4} \]

Here, \( \kappa \) is the cavity damping rate, \( \omega_c \) the cavity resonance frequency, and \( a_{\text{in}} \) \( (\alpha_{\text{in}}) \) the noise with the correlators given in Eq. (A.1) \( \text{(Eq. (A.2))} \) and zero mean, i.e., \( \langle a_{\text{in}} \rangle = \langle \hat{a}_{\text{in}} \rangle = 0 \) and \( \langle a_{\text{in}}^* \rangle = \langle \alpha_{\text{in}}^* \rangle = 0 \). This differential equation can be formally solved: \( \hat{a}(t) = \sqrt{\kappa} \int_{t'}^t dt' \exp \left( i \omega_c + \kappa/2 \right) \hat{a}_{\text{in}}(t') \) and similarly for the semi-classical equation. Interestingly, we then find that true quantum fluctuations lead to

\[ \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = 0 \text{ and } \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = 1 \tag{A.5} \]

while the semi-classical correlations lead to

\[ \langle \alpha^*(t) \alpha(t) \rangle = \langle \alpha(t) \alpha^*(t) \rangle = \frac{1}{2}. \tag{A.6} \]

Thus, the method of mimicking quantum noise leads to a photon occupation of half a photon even in the absence any source. This is, of course, not the case for true quantum fluctuations. The semi-classical model can still be very useful when the consequences of this unavoidable symmetry in the classical noise correlators are negligible \( \text{(e.g., in the large excitation limit)} \) or only symmetrized expectation values are calculated. In an optomechanical system a completely analogue argument holds for the mechanical oscillator. Notably, if really large values of optomechanical coupling \( g_0 \) are investigated then already this vacuum photon occupation of \( 1/2 \) significantly interacts with the mechanical oscillator and changes the vacuum phonon number occupation of \( 1/2 \). For instance, in Fig. 2.4, where we compare the results of a few quantum jump trajectories with the semi-classical model, the comparison could be improved if one would correct for this “vacuum optomechanical interaction” which is of purely numerical origin.

In Sec. 2.9, we introduced thermal noise on the mechanical oscillator in addition to the mimicked quantum noise. In this case the noise correlators are modified to contain also the thermal bath occupancy \( n_{\text{th}} \). Regarding the implementation this rescales the variance of the involved random number distributions, as can be seen from the new noise correlators.

Due to the noise in our model we typically observed no dependence on the initial state. Only when studying the classical-to-quantum crossover, cf. Sec. 2.7, we found an “artificial” dependence when approaching the classical (noiseless) limit. This originates from the weak noise strength in combination with a bistable system: Long simulation times are necessary before weak noise drives a transition from one to the other stable state (a potential barrier has to be crossed). Whenever simulation times are not sufficiently long to average over many
A. Numerical methods

of these transitions, then averaging many trajectories with different initial conditions seems to lead to different results than the time average of a single trajectory (which is simply not sufficiently long). This has been taken care of when interpreting our results and was actually a limiting factor in simulating the crossover to the classical limit. The required simulation times diverge dramatically, especially if we want to collect statistical data on the transitions and not only explore the potential landscape. The fully classical noiseless limit is described by a deterministic differential equation and a full study has to include different initial conditions.

A.3 Unitary time evolution of nonlinear equations of motion

In Ch. 4 we do not deal with an open quantum system. Although one could include driving or dissipation to the environment, we focused on the closed system that can be fully described by a Hamiltonian. Complexity entered via a nonlinear term, such that, at the end, we had to solve a nonlinear Schrödinger equation or the corresponding classical equations of motion, which are all of the type

\[ i\dot{\Psi} = (H + V(\Psi))\Psi, \]

on a lattice. Here, \( H \) and \( V(\Psi) \) are matrices containing all linear and nonlinear terms, respectively. This is then a set of differential equations that can be solved using standard numerical methods for differential equations, such as the Euler method, cf. A.1, or Runge-Kutta integration. While these methods work well (without fine-tuning) for the linear problem, i.e. for \( V = 0 \), we faced numerical instabilities in any other case. In our analysis we observed that this was triggered by the loss of normalization: Because the Euler method, \( \Psi(t + \delta t) \approx (1 - i(H + V)\delta t)\Psi(t) \), is not unitary, it is possible that after each timestep \( \delta t \) the total intensity \( |\Psi|^2 \) on the lattice changes slightly (although there is no drive or dissipation). In the nonlinear case this effect adds up, resulting in a rapid divergence of \( |\Psi|^2 \). This is a fundamental problem that is not simply solved by choosing smaller timesteps or using a better-performing (still non-unitary) integration method.

A straight-forward solution is to enforce a unitary time evolution by using a slightly different approximation than the Euler method, i.e.,

\[
\Psi(t + \delta t) = e^{-i(H+V)\delta t/2} e^{i(H+V)\delta t/2} \Psi(t) \\
\approx \left(1 - i(H + V)\frac{\delta t}{2}\right) \left(1 + i(H + V)\frac{\delta t}{2}\right)^{-1} \Psi(t). \tag{A.7}
\]

However, since \( V(\Psi(t)) \) has to be updated at each timestep, this approach requires to calculate the inverse of a non-diagonal matrix at each timestep. This is certainly not cost-efficient. Fortunately, the nonlinear \( V(\Psi) \) in our particular problem in Ch. 4 is diagonal, i.e., the nonlinearity in Eq. (4.7) depends only on the local intensity \( |\Psi_j|^2 \). Thus, we can
settle for a pragmatic approach in which we split our problem into two steps for each \( \delta t \), one for the linear part and another for the nonlinear one. Note, however, that this separate treatment introduces another level of approximation since \( H \) and \( V \) do not necessarily commute. Let us now explain this approach in detail: First we perform the linear time evolution only:

\[
\Psi(t + \delta t) \approx \left(1 - iH\frac{\delta t}{2}\right)\left(1 + iH\frac{\delta t}{2}\right)^{-1}\Psi(t),
\]

where the inverse matrix \( \left(1 + iH\frac{\delta t}{2}\right)^{-1} \) has to be determined only once (since \( H \) is constant) and can then be reused at all timesteps. In fact, the full block \( \left(1 - iH\frac{\delta t}{2}\right)\left(1 + iH\frac{\delta t}{2}\right)^{-1} \) can be pre-computed and at each timestep only a single matrix-vector multiplication has to be performed. Second, the nonlinear time evolution is done afterwards:

\[
\Psi(t + \delta t) \approx \left(1 - iV\frac{\delta t}{2}\right)\left(1 + iV\frac{\delta t}{2}\right)^{-1}\tilde{\Psi}(t + \delta t).
\]

The inverse matrix \( \left(1 + iV\frac{\delta t}{2}\right)^{-1} \) is trivial, since it is the inverse of a diagonal matrix only. The matrix \( \left(1 - iV\frac{\delta t}{2}\right)\left(1 + iV\frac{\delta t}{2}\right)^{-1} \) can be set up directly and the multiplication with the vector \( \tilde{\Psi} \) actually requires only the multiplication-by-entry of two vectors and not a real matrix-vector product.

Using this approach for our numerical calculation, we find the results shown in Figs. 4.6 and 4.7.

A.4 Summary of programming tools

During the work on this thesis, we have used a large number of different scientific software, trying to find the best balance between convenient usage and computational speed for each specific problem. Without going into too much detail, we here want to briefly state which software tools were used in the different parts of this thesis.

In Ch. 2 the numerical challenge was highest. Therefore, we developed our own C code and compiled it with the GNU compiler collection (gcc) using optimization flags.\(^3\) This code was accessed via a yorick interface, which was then also used to postprocess and plot the data. Note that these simulations were mostly run on our small "cluster" of seven desktop computers, each with 8 cores, which allowed to compute several trajectories in parallel and run the code without disruption for up to a few weeks. The stochastic Langevin equations were also solved using a C and yorick combination, although there the speed up of the simulation was not crucial to make the problem accessible. yorick is an interpreted programming language, that has a very useful array-syntax and allows to produce high-quality figures at the same time. We used gnuplot to fit data and the plot program of Mac OS for some graphics.

In Ch. 3 the complexity of the required numerical simulations was much lower, since the model only contained one oscillator. No problem-specifically optimized program to solve the corresponding master equation was necessary. Instead, we used the freely available, python-
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Based, open-source computational toolbox for quantum optics, qutip [222, 223]. In particular, we used Python 3.4.4 with the package qutip 3.1.0. All Wigner densities presented in this thesis (also the ones in the Introduction) were computed with qutip. This toolbox is highly recommendable for such problems, since it contains simple commands to span Hilbert spaces, standard routines to solve Schrödinger or master equations, and many other useful pre-defined standard analysis tools such as, e.g., routines to calculate the Wigner density from an arbitrary density matrix. In combination with the standard packages of python for numerical simulations (mainly numpy and scipy) and the simple production of figures via the package matplotlib, this results in an excellent working environment. Let us note that, depending on the Hilbert space size, different routines to calculate the Wigner density should be applied. In particular, for large Hilbert spaces (approximately more than 50 states) we had to specify method='fft' to achieve appropriate results, although this takes more computation time. For further details we refer to the qutip documentation. Another peculiarity is the calculation of the spectra presented in Ch. 3. In contrast to all other computations in this chapter, they were obtained with MATLAB instead of python. We determined the spectra applying the quantum regression theorem, as described in detail in Ref. [224]. In this case we used MATLAB due to its straight-forward parallelization of for-loops, which lead to a significant speed-up.

Furthermore, the results presented in Ch. 4 were obtained using MATLAB, which, similar to python, provides a convenient array-syntax and well-documented numerical and plotting routines.

Throughout the whole thesis, some analytical calculations were performed and/or assisted using Mathematica.

\footnote{Of course, this can also be done in python - we just happened to know already how it is done in MATLAB.}
BIBLIOGRAPHY


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List of publications

During the work for this thesis, the following articles have been published in refereed journals:

- Talitha Weiss, Stefan Walter, and Florian Marquardt
  *Quantum-coherent phase oscillations in synchronization*
  *Phys. Rev. A 95, 041802(R) (2017)*

- Talitha Weiss, Andreas Kronwald, and Florian Marquardt
  *Noise-induced transitions in optomechanical synchronization*
  *New J. Phys. 18, 013043 (2016)*

Previous and additional work (not part of this thesis) has been published in refereed journals or as a preprint:

- Carlos Navarrete-Benlloch, Talitha Weiss, Stefan Walter, Germán J. de Valcárcel
  *General linearized theory of quantum fluctuations around arbitrary limit cycles*

- Talitha Weiss and Andreas Nunnenkamp
  *Quantum limit of laser cooling in dispersively and dissipatively coupled optomechanical systems*

- Talitha Weiss, Christoph Bruder, and Andreas Nunnenkamp
  *Strong-coupling effects in dissipatively coupled optomechanical systems*
  *New J. Phys. 15, 045017 (2013)*
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