Since the early stages of quantum theory, physicists were debating over the following observation. While quantum superpositions are abundant in quantum systems, we essentially never observe superpositions on the macroscopic scale. One possible explanation for this apparent paradox was given by decoherence theory, i.e. the destruction of quantum superposition by interaction with the environment. In this worksheet, you will be asked to derive the timescale over which the quantum superpositions of a quantum harmonic oscillator are destroyed due to its interaction with a low temperature reservoir. The paperwork is about decoherence.

Exercise 31: Decoherence time

We consider a quantum harmonic oscillator interacting with a zero-temperature thermal reservoir. The time evolution of the density operator of this system in the interaction picture is given by the following master equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \frac{\gamma}{2}\left(2a\rho a^\dagger - \left\{a^\dagger a, \rho\right\}\right). \tag{1}$$

The solution of master equations for infinite-dimensional systems is challenging. A typical approach for harmonic oscillators makes use of phase space pseudo-probability distributions, like the Wigner distribution, that you have encountered before. As for genuine probability distributions, one can introduce characteristic functions for these probability distributions. An example is given by

$$X_N(\eta, t) = Tr\left(\rho(t)\exp(\eta a^\dagger)\exp(-\eta^* a)\right). \tag{2}$$

In the following you will derive the time scale over which superpositions of the harmonic oscillator are destroyed by the interaction with the reservoir. The following identities might be useful: $[a^\dagger, f(a,a^\dagger)] = -\frac{\partial f(a,a^\dagger)}{\partial a}$, $[a, f(a,a^\dagger)] = \frac{\partial f(a,a^\dagger)}{\partial a^\dagger}$, $[a^\dagger a, \exp(\eta a^\dagger)\exp(-\eta^* a)] = \eta a^\dagger \exp(\eta a^\dagger)\exp(-\eta^* a) + \eta^* \exp(\eta a^\dagger)\exp(-\eta^* a)a$.

1) Make use of the master equation given in eq. 1 to derive a first oder partial differential equation for the time evolution of the characteristic function given in eq. 2. (Hint: the equation will take the following form: $\frac{\partial X_N(\eta,t)}{\partial t} = -\frac{\gamma}{2}\left[\eta\frac{\partial X_N(\eta,t)}{\partial \eta} + \eta^*\frac{\partial X_N(\eta,t)}{\partial \eta^*}\right]$)

2) Show that $X_N(\eta,t) = X_N\left(\eta\exp\left(-\frac{\gamma}{2}t\right), 0\right) = X_N(\eta(t), 0)$ is a solution of the partial differential equation obtained above.

3) Assuming that the harmonical oscillator is initially in a superposition of coherent states $|\alpha_i\rangle$, calculate $X_N(\eta, 0)$, i.e. the initial condition.

4) Based on your previous results, what is the expression for $X_N(\eta, t)$? Assuming that the state is in a superposition of the two coherent states $|\pm\alpha\rangle$, how would the density matrix look like?

5) Considering the long-time limit $\gamma t \gg 1$, what is the time scale over which the off-diagonal terms (coherences) vanish?

Exercise 32: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

Observing the Progressive Decoherence of the Meter in a Quantum Measurement
M. Brune *et al.*
Phys. Rev. Lett. **77**, 4887 (1996).

Decoherence of quantum superpositions through coupling to engineered reservoirs
M. Myatt *et al.*
Nature **403**, 269 (2000).