Friedrich-Alexander Universität Erlangen-Nürnberg

WS 2012/2013

Institut für Theoretische Physik II
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14. Übungsblatt Many-body physics with ultra-cold atomic gases

29.02.2013
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14.1: Fermi-Hubbard model

The Fermi Hubbard model on a simple cubic (i.e. bipartite) lattice is a rather symmetric model with many discrete symmetries that can be used to map parts of the phase diagram onto other parts.

As a first transformation we consider the *particle-hole* transformation:

$$d^\dagger_{\mathbf{i}\sigma} = (-1)^i c_{\mathbf{i}\sigma}$$

Here $c_{\mathbf{i}\sigma}$ denotes the annihilation operator of a fermionic particle with spin σ on site $\mathbf{i}$ and $d^\dagger_{\mathbf{i}\sigma}$ is the creation operator of the corresponding hole: Instead of considering particles on top of the vacuum state we now consider holes in a perfect band-insulator. In higher dimension we use the shorthand $\mathbf{i} = (i_x, i_y, i_z)$ and $i = i_x + i_y + i_z$.

(a) Express the number operator of holes $n^h_{i\sigma}$ in terms of the number operator of particles $n^p_{i\sigma}$.

(b) Verify that the kinetic energy takes exactly the same form for the hole operators d than for the original operators c. What is the effect of the $(-1)^i$ term?

(c) Show that a slightly modified interaction term $\hat{H}_{\text{IA}} = \sum_i U(n^p_{\mathbf{i}\uparrow} - 1/2)(n^p_{\mathbf{i}\downarrow} - 1/2)$ also remains invariant under the particle-hole transformation. Why does this interaction still describe the same system as the original Hubbard term?

(d) Where was the bipartite nature of the lattice used in the above transformation?

Another important transformation is the so-called *partial particle-hole* or *Lieb-Mattis* transformation. As the name suggests, this transformation consists of a particle-hole transformation for the down spins while the up-spins remain unchanged:

$$d^\dagger_{\mathbf{i}\uparrow} = c^\dagger_{\mathbf{i}\uparrow} \qquad d^\dagger_{\mathbf{i}\downarrow} = (-1)^i c_{\mathbf{i}\downarrow}$$

Obviously, this transformation also leaves the kinetic energy unchanged.

(e) How does the above interaction term $\hat{H}_{\text{IA}}$ transform into the new basis?

(f) How would i)a Mott insulating state with maximum spin disorder and ii)a perfect N?el state look in the new basis? (Assume the large U limit.)

(g) How do the three components of the magnetization ($m^c_{z\mathbf{i}} = n^c_{\mathbf{i}\uparrow} - n^c_{\mathbf{i}\downarrow}, \quad m^c_{+\mathbf{i}} = c^\dagger_{\mathbf{i}\uparrow} c_{\mathbf{i}\downarrow}, \quad m^c_{-\mathbf{i}} = c^\dagger_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow}$) transform under the partial particle-hole transformation?

14.2: General Questions: Fermions, cold gases in optical lattices

Answer in your own words.

(a) Discuss the temperature dependence of the gap and the chemical potential of a two-component Fermi gas as a function of $1/k_F a$ as one goes from the BEC to the BCS limit.

(b) What is definition of the superfluid density? What is the definition of the condensate fraction?

(c) Describe the unitary Fermi gas.

(d) Fig. 1 shows the grand-canonical phase diagram of a spin-imbalanced Fermi gas with attractive interactions in one dimension. Use LDA to predict the shell structure of such a system in a harmonic trap.

(e) What properties define a Mott insulator?

(f) Sketch the phase diagram of the Bose-Hubbard model.

(g) Sketch the quasi-momentum distribution in the BHM of a MI in the atomic limit and at $U = 0$.

(h) Describe the ground-state of a MI in the Bose-Hubbard model in the limit $U/J \gg (U/J)_{crit}$ large.

(i) What quantity is actually measured in single-site resolution experiments for bosons in optical lattices?

(j) For atoms in an optical lattice, what is the difference between a time-of-flight measurement and band-mapping?

(k) Sketch the phase diagram of the Fermi-Hubbard model in 3D. How does the phase boundary of the antiferromagnet depend on U? Why is that?

(l) What are the low-energy excitations in the half-filled Fermi-Hubbard model? Which Hamiltonian describes the low-energy physics?

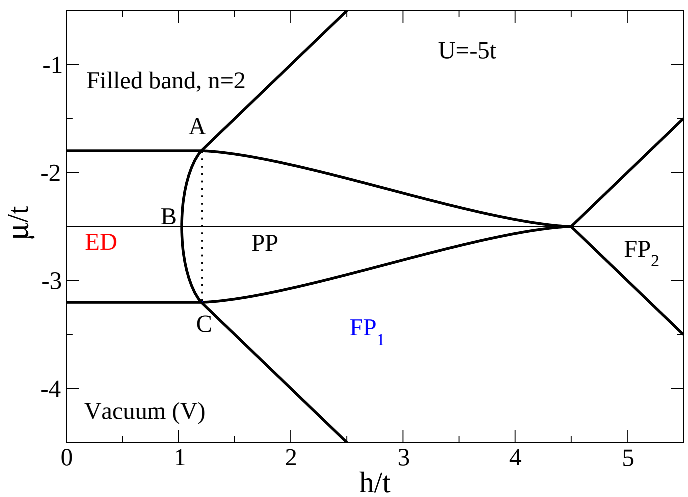

Abbildung 1: The figure shows the phase diagram of a two-component Fermi gas with attractive interactions in a one-dimensional lattice system, described by the Hubbard model with negative U. The phases are: (i) ED: equal density, $N_\uparrow = N_\downarrow$, (ii) FP_1: fully polarized, $N_\downarrow = 0$, $N_\uparrow > 0$ (with filling $n = (N_\uparrow + N_\downarrow)/L < 1$). (iii): PP: partially polarized phase, $N_\downarrow \neq N_\uparrow$, (iv) FP_2: $N_\downarrow = 0$, $N_\uparrow > 0$, with $N_\uparrow/L = 1$, where L is the number of lattice sites.