In the lecture the effective Hamiltonian describing the Jaynes-Cummings model in the limit of large detuning was introduced. In the course of this exercise sheet, you will be asked to derive it from the exact Jaynes-Cummings model. The paper work is about collapse and revivals, and the creation of Schrödinger cat states.

Exercise 26: Large detuning Jaynes-Cummings Hamiltonian

The Jaynes-Cummings model is one of the most important models in quantum optics. It describes the interaction of a two-level system interacting with a single mode of a quantized electical field. The Hamiltonian of the single mode of the electric field and the two level system will be given as

$$H_0 = \hbar\omega a^\dagger a + \frac{\hbar\omega_0}{2}\sigma_z, \tag{1}$$

where ω denotes the mode of the electric field, $\hbar\omega_0 = E_2 - E_1$ denotes the energy difference between the two levels of the atom, $a, a^\dagger$ are the ladder operators of the harmonical oscillator and σ_z is the Pauli matrix. Here we ignored the ground state energy of the electric field. We assume that the interaction between the electric field and two-level atom is described by the interaction of the field with the dipole moment of the atom $\vec{d}$ via the Hamiltonian

$$H_{int} = -q\vec{d} \cdot \vec{E}(\vec{r}, t). \tag{2}$$

By expressing this operator in terms of the harmonic oscillator and Pauli operators, transforming it to the interaction picture and ignoring rapidly oscillating terms (rotating-wave approximation) one obtains the following interaction-picture Hamiltonian

$$H_{int}(t) = \hbar g \left(\sigma_- a^\dagger e^{i\Delta t} + \sigma_+ a e^{-i\Delta t}\right), \tag{3}$$

where $\Delta = \omega - \omega_0$ describes the detuning between the frequency of the electric field mode ω and the energy splitting of the two-level system ω_0. In the following, you will derive the effective Hamiltonian

$$H_{eff} = \hbar\chi \left[\sigma_+\sigma_- + a^\dagger a \sigma_+\right] \quad \text{with } \chi = \frac{g^2}{\Delta} \tag{4}$$

by using time dependent perturbation theory. To that end, consider the expansion of the time evolution operator $U(t, t_0)$ (Dyson series) to second order, i.e.

$$U(t, 0) \approx \mathbb{1} - \frac{i}{\hbar}\int_0^t dt' H(t') - \frac{1}{\hbar^2}\int_0^t dt' H(t') \int_0^{t'} dt'' H(t'') \tag{5}$$

1) Start by calculating the first order contribution in $1/\hbar$ to the Dyson series in the interaction picture. (Hint: That means use $H_{int}(t)$ instead of H in the Dyson series given in eq. 5)
2) Use your result for the first order contribution to write down the expression for the second order. Use the observation that the two level system can neiter be excited above the excited state nor deexcited below the ground state ($\sigma_\pm^2 = 0$) to simplify your result.
3) Before performing the final integration, further simplify the integrand, by discarding all terms that are oscillatory and only keep the constant contributions. Now perform the final integration.
4) Compare the the first order and second order contribution. Find arguments to neglect the first order compared to the second order contributions.
5) Write down the final approximation for the time evolution operator. Which time independent Hamiltonian would lead to the same dynamics?

Exercise 27: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

Observation of quantum collapse and revival in a one-atom maser
G. Rempe, H. Walther, and N. Klein
Phys. Rev. Lett. **58**, 353-356 (1987)

A "Schrödinger Cat" Superposition State of an Atom
C. Monroe, D. M. Meekhof, B. E. King, D. J. Wineland
Science **272**, 1131 (1996)

A Schrödinger cat living in two boxes
C. Wang *et al.*
Science **352**, 1087 (2016).