"Quantum optics"

WS 2016-2017 Worksheet 8

The aim of this worksheet is to gain a better understanding of the displacement operator. The papers are devoted to the Casimir effect.

Exercise 20: Displacement operator

To better understand the displacement operator, let us start with a study of the translation operator in quantum mechanics.
Consider the operator $\hat{T}(\lambda) = e^{-\frac{i\lambda\hat{p}}{\hbar}}$ where $\hat{p}$ is the momentum operator with the commutation relation $[\hat{x}, \hat{p}] = i\hbar$.

1) Show that $\hat{T}(\lambda)$ is unitary: $\hat{T}^\dagger(\lambda) = \hat{T}^{-1}(\lambda) = \hat{T}(-\lambda)$.

2) Calculate the commutator $[\hat{x}, \hat{T}(\lambda)]$ and show that it implies $\hat{x}\hat{T}(\lambda) = \hat{T}(\lambda)(\hat{x} + \lambda)$. Show that $\hat{T}(\lambda)\hat{T}(\mu) = \hat{T}(\lambda + \mu)$.

3) Consider the eigenvalue equation $\hat{x}|x\rangle = x|x\rangle$. Use the above relations to show that $\hat{x}\hat{T}(\lambda)|x\rangle = (x + \lambda)\hat{T}(\lambda)|x\rangle$. Interpret this result.

4) By setting $|x\rangle = \hat{T}(x)|0\rangle$, show that $\hat{T}(\lambda)|x\rangle = |x+\lambda\rangle$. What is the action of the translation operator on an arbitrary state $|\psi\rangle$ in the coordinate representation $\psi(x) = \langle x|\psi\rangle$?

5) Show that $\hat{T}^\dagger(\lambda)\hat{x}\hat{T}(\lambda) = \hat{x} + \lambda$ and compare this with the properties of the displacement operator for the coherent state. What is the form of the translation operator in momentum (i.e. the operator $\hat{\mathfrak{T}}(\lambda)$ with the property $\hat{\mathfrak{T}}(\lambda)|p\rangle = |p + \lambda\rangle$)?

6) Evaluate the product of $e^{\frac{2i\alpha''\hat{X}}{\hbar}}$ and $e^{-\frac{2i\alpha'\hat{P}}{\hbar}}$ and show that it is given by $e^{\alpha a^\dagger - \alpha^* a}$ (apart from a phase factor), with $\alpha = \alpha' + i\alpha''$. Here $\hat{X}$ and $\hat{P}$ are the dimensionless position and momentum operators $\hat{X} = \frac{a+a^\dagger}{2}$, $\hat{P} = \frac{a-a^\dagger}{2i}$.

Exercise 21: Paper-Work

Find the following articles online. Read and understand them! Try to answer the following questions for each:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

Precision Measurement of the Casimir Force from 0.1 to 0.9 μm
U. Mohideen and Anushree Roy
Phys. Rev. Lett. **81**, 4549–4552 (1998)

Measurement of the Casimir Force between Parallel Metallic Surfaces
G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso
Phys. Rev. Lett. **88**, 041804 (2002)

Observation of the dynamical Casimir effect in a superconducting circuit
C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, P. Delsing
Nature **479**, 376 (2011)