

- IDENTICAL PARTICLES -

Saying that a system is composed of N identical particles is equivalent to saying that there is no observable capable of distinguishing between them.

For example, consider two particles, say "a" and "b", of the same kind (e.g., two electrons, or two photons, etc).

Each of these particles possess a CSCO, namely a complete set of commuting observables A, B, \dots, Z , such that the state $|a, b, \dots, z\rangle$ is uniquely determined.

For simplicity, let us denote with ξ the set of eigenvalues of $\hat{A}, \hat{B}, \dots, \hat{Z}$:

$$\xi = (a, b, \dots, z)$$

and use the shorthand

$$|\xi\rangle = |a, b, \dots, z\rangle$$

The Hilbert space \mathcal{H} of the system formed by the two particles is simply

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$$

and it is spanned by the basis vectors

$$|\xi_1, \xi_2\rangle_{ab} = |\xi_1\rangle_a |\xi_2\rangle_b = |\xi_1\rangle_a \otimes |\xi_2\rangle_b$$

All equivalent ways to write the same vector

where from now on we adopt the convention that in the ket

$$|\xi_1, \xi_2\rangle_{ab}$$

The first label (here ξ_1) always refers to particle "a" and the second one (here ξ_2), to particle "b". Therefore, from now on we drop the labels "a" and "b".

Now, consider the two different states:

$$|\psi_{12}\rangle \equiv |\xi_1, \xi_2\rangle$$

$$|\psi_{21}\rangle \equiv |\xi_2, \xi_1\rangle$$

When the two-particle system is prepared in the state $|\psi_{12}\rangle$, particle a is characterized by the set of values ξ_1 and particle b by the values ξ_2 . Vice versa for the system prepared in the state $|\psi_{21}\rangle$.

Suppose to make a measurement on the system and to find the values ξ_1 for one particle and ξ_2 for the other one. If the particles are indistinguishable, then we do not know if the state was $|\psi_{12}\rangle$ or $|\psi_{21}\rangle$ or even a linear combination

$$|\psi\rangle = c_1|\psi_{12}\rangle + c_2|\psi_{21}\rangle$$

of the two.

The degeneracy of $|Y_{12}\rangle$ and $|Y_{21}\rangle$ is called the "exchange degeneracy".

Let \hat{P} be the permutation operator defined by:

$$\hat{P}|\xi_1, \xi_2\rangle = |\xi_2, \xi_1\rangle$$

or,
$$\hat{P}|Y_{12}\rangle = |Y_{21}\rangle$$

by definition,

$$\hat{P}^2 = \hat{I}$$

(\hat{X}_a and \hat{X}_b are the same operator but acting on different particles)

If $\hat{A} \equiv \hat{X}_a \otimes \hat{I}_b$ and $\hat{B} \equiv \hat{I}_a \otimes \hat{X}_b$ such that:

1.3a)
$$\hat{A}|\xi_1, \xi_2\rangle = (\hat{X}_a|\xi_1\rangle)|\xi_2\rangle = \xi_1|\xi_1, \xi_2\rangle$$

1.3b)
$$\hat{B}|\xi_1, \xi_2\rangle = |\xi_1\rangle(\hat{X}_b|\xi_2\rangle) = \xi_2|\xi_1, \xi_2\rangle$$

Multiply 1.3a) by \hat{P} from left:

$$\hat{P}(\hat{A}|\xi_1, \xi_2\rangle) = \xi_1 \hat{P}|\xi_1, \xi_2\rangle = |\xi_2, \xi_1\rangle$$

$$= (\hat{P}\hat{A}\hat{P}^{-1})(\hat{P}|\xi_1, \xi_2\rangle)$$

$$= (\hat{P}\hat{A}\hat{P}^{-1})|\xi_2, \xi_1\rangle$$

$$\Leftrightarrow (\hat{P}\hat{A}\hat{P}^{-1})|\xi_2, \xi_1\rangle = \xi_1|\xi_2, \xi_1\rangle$$

Comparing 1.3b) with 2.3) we infer that:

$$\hat{P} \hat{A} \hat{P}^{-1} = \hat{B}$$

$$\Leftrightarrow \hat{P} (\hat{X}_a \otimes \hat{I}_b) \hat{P}^{-1} = \hat{I}_a \otimes \hat{X}_b$$

So, the operator \hat{P} exchange the labels a and b.

* Example *

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + \frac{\hat{P}_2^2}{2m} + V_{\text{pair}}(|\underline{r}_1 - \underline{r}_2|) + V(\underline{r}_1) + V(\underline{r}_2)$$

Clearly: $\hat{H}(\hat{r}_1, \hat{p}_1; \hat{r}_2, \hat{p}_2) = \hat{H}(\hat{r}_2, \hat{p}_2; \hat{r}_1, \hat{p}_1)$

$$\Rightarrow [\hat{H}, \hat{P}] = 0 \Rightarrow \hat{P} \text{ is a constant of motion} \Leftrightarrow$$

\Leftrightarrow a state maintains its exchange symmetry upon time evolution because from $\hat{P}^2 = \hat{I}$ it follows

$$\text{that } \hat{P}^2 - \hat{I} = 0 \Leftrightarrow (\hat{P} - \hat{I})(\hat{P} + \hat{I}) = 0 \Rightarrow$$

\Rightarrow the eigenvalues of \hat{P} are ± 1

Therefore, if $|\psi(t=0)\rangle : \hat{P}|\psi(t=0)\rangle = \pm |\psi(t=0)\rangle$, then this is true for any time $t > 0$.

More generally, in a system of N identical particles, any observable \hat{A} must commute with \hat{P} .

$$[\hat{A}, \hat{P}] = 0$$

If this were not true, then we could distinguish the particles.

Consider now 3 particles "a", "b" and "c",

with

$$H = H_a \otimes H_b \otimes H_c$$

and consider the normalized states

$$|\xi_i, \xi_j, \xi_k\rangle \equiv |i, j, k\rangle$$

↑
short hand

with $i \neq j \neq k$ and ^{assume that} each ξ_i can take only the three values

$$\xi_i = 1, 2, 3.$$

The 6 states we are considering are:

$$\left\{ \begin{aligned} |\psi_S\rangle &= \frac{1}{\sqrt{6}} [|1123\rangle + |1132\rangle + |1231\rangle + |1213\rangle + |1312\rangle + |1321\rangle] \\ |\psi_A\rangle &= \frac{1}{\sqrt{6}} [|1123\rangle - |1132\rangle + |1231\rangle - |1213\rangle + |1312\rangle - |1321\rangle] \\ |\psi_{M_1}\rangle &= \frac{1}{2\sqrt{3}} [2|1123\rangle - |1132\rangle + 2|1213\rangle - |1231\rangle - |1312\rangle - |1321\rangle] \\ |\psi_{M_2}\rangle &= \frac{1}{2} [|1132\rangle - |1231\rangle + |1312\rangle - |1321\rangle] \\ |\psi_{M_3}\rangle &= \frac{1}{2\sqrt{3}} [2|1123\rangle + |1132\rangle - 2|1213\rangle - |1231\rangle - |1312\rangle + |1321\rangle] \\ |\psi_{M_4}\rangle &= \frac{1}{2} [|1132\rangle + |1231\rangle - |1312\rangle - |1321\rangle] \end{aligned} \right.$$

We assume the normalization

(23=6)

$$\langle i' j' k | i j k \rangle = \delta_{i'i} \delta_{j'j} \delta_{k'k}$$

clearly

$$\hat{P}|\psi_S\rangle = |\psi_S\rangle$$

$$\hat{P}|\psi_A\rangle = \pm |\psi_A\rangle \quad \begin{cases} + & \text{even permutation} \\ - & \text{odd permutation} \end{cases}$$

There are $N! = 3!$ possible permutations in S_N (permutation group of N elements)

However, $\hat{P}|\psi_{m_a}\rangle \neq \lambda |\psi_{m_a}\rangle \quad (a=1,2)$

but $\hat{P}(c_1|\psi_{m_1}\rangle + c_2|\psi_{m_2}\rangle) = c_1|\psi_{m_1}\rangle + c_2|\psi_{m_2}\rangle$

This means that the set $\{|\psi_{m_1}\rangle, |\psi_{m_2}\rangle\}$ is stable against permutations.

The same is true for the set $\{|\psi_{m_1}\rangle, |\psi_{m_2}\rangle\}$

The states $|\psi_{m_a}\rangle$ and $|\psi_{m_b}\rangle$ are said to have Mixed symmetry.

If $|\psi_{m_1}\rangle$ were physically realizable, the same physical state would be represented by the essentially different states $|\psi_{m_1}\rangle$ and $\hat{P}|\psi_{m_1}\rangle$

So, The Hilbert space \mathcal{H}_3 of a system of three identical particles can be decompose as the direct sum of four (4) subspaces:

$$\mathcal{H}_3 = \mathcal{H}_S \oplus \mathcal{H}_A \oplus \underbrace{\mathcal{H}_M \oplus \mathcal{H}_{M'}}_{\equiv \mathcal{H}_M \text{ mixed sym}}$$

\uparrow totally sym. \uparrow totally antisym.

the decomposition $\mathcal{H}_M = \mathcal{H}_M \oplus \mathcal{H}_{M'}$ is not unique

Note: From $[\hat{A}, \hat{P}] = 0$ for any observable \hat{A} , it follows that the matrix elements between $\mathcal{H}_S, \mathcal{H}_A$ and \mathcal{H}_M are zero. This is called a superselection rule.

• Moreover, any eigenvalue of \hat{A} in \mathcal{H}_M is at least doubly degenerate (exchange degeneracy)

• If $N=2$, we can only have $\mathcal{H}_2 = \mathcal{H}_S \oplus \mathcal{H}_A$
 $\underbrace{\hspace{2em}}_{3 \text{ dim}}$ $\underbrace{\hspace{2em}}_{1 \text{ dim}}$

• If $N > 3$, in general we have:

$$\mathcal{H}_N = \mathcal{H}_S \oplus \mathcal{H}_A \oplus \mathcal{H}_{M_1} \oplus \mathcal{H}_{M_2} \oplus \dots$$

Each \mathcal{H}_{M_i} has a complicated symmetry and furnish higher-dimensional representations of S_N .

*A very important result is the symmetrization Principle:

- The pure states of a system of identical particles must be totally symmetric or antisymmetric under the exchange of any two of them.

Why? Well, if we admit that a physical system of N identical particles possesses some CSCO $\{A_1, A_2, \dots, A_r\}$, then its pure state cannot belong to any subspaces of mixed symmetry. In fact, if a_1, \dots, a_r are the eigenvalues of $\hat{A}_1, \dots, \hat{A}_r$, then the state $|a_1, \dots, a_r\rangle$ is essentially unique. But if $|a_1, \dots, a_r\rangle \in \mathcal{H}_{\mu_i}$, then we can find some permutation P such that

$$\hat{P}|a_1, \dots, a_r\rangle \neq e^{i\alpha}|a_1, \dots, a_r\rangle$$

for all α , and at the same time $\hat{P}|a_1, \dots, a_r\rangle$ is an eigenstate of $\hat{A}_1, \dots, \hat{A}_r$ with eigenvalues a_1, \dots, a_r , thus leading to a contradiction.

* In relativistic quantum theory is possible to establish the celebrated spin-statistic theorem (Pauli, 1941):

- The pure states of a system of identical particles are totally symmetric (antisymmetric) if their spin is an integer (half-odd integer).

Note: identical particles with totally symmetric wave functions obey Bose-Einstein statistics (these particles are called bosons), while particles with totally antisymmetric wave functions obey the Fermi-Dirac statistics (these particles are called Fermions).

Experimental data confirms that:

Fermions with spin $\frac{1}{2}$: electrons, nucleons, He^3 , quarks, Muon, Tau + neutrinos

Bosons with spin 0 or 1: photons, He^4 , mesons (π , K , J/ψ , ...)

For Fermions is valid the Pauli exclusion principle: "Two fermions cannot be in the same state".

Proof: if \hat{P}_{ij} exchange ξ_i with ξ_j , then for a system of N Fermions:

$$\hat{P}_{ij} |\xi_1, \dots, \xi_i, \dots, \xi_j, \dots, \xi_N\rangle = - |\xi_1, \dots, \xi_j, \dots, \xi_i, \dots, \xi_N\rangle$$

$$\Rightarrow |\xi_1, \dots, \xi_i, \dots, \xi_i, \dots, \xi_N\rangle = 0 \text{ if } \xi_i = \xi_j.$$

Case study - Two-electron systems -

Two electrons, say "1" and "2".

$$H = H_1 \otimes H_2$$

H_i is span by $\{|+\rangle_i, |-\rangle_i\}$ ($i=1,2$)

where $\hat{S}_{i,z} |\pm\rangle_i = \pm \frac{\hbar}{2} |\pm\rangle_i$

there are 4 independent product states

$$|+, +\rangle$$

$$|+, -\rangle$$

$$|-, +\rangle$$

$$|-, -\rangle$$

The states $|+, +\rangle$ and $|-, -\rangle$ are symmetric.

$$\hat{P} |+, +\rangle = |+, +\rangle$$

while $|+, -\rangle$ and $|-, +\rangle$ get mixed.

$$\hat{P} |+, -\rangle = |-, +\rangle$$

$$\hat{P} |-, +\rangle = |+, -\rangle$$

Then we can build the symmetric and antisymmetric linear combinations:

$$\text{sym: } \frac{1}{\sqrt{2}} (|+, -\rangle + |- , +\rangle)$$

$$\text{antisym: } \frac{1}{\sqrt{2}} (|+, -\rangle - |- , +\rangle)$$

In summary, we can decompose $\mathcal{H}_1 \otimes \mathcal{H}_2$ as:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \underbrace{\mathcal{H}_S}_{\substack{3\text{-dim} \\ \text{sym. space}}} \oplus \underbrace{\mathcal{H}_A}_{\substack{1\text{-dim} \\ \text{antisym space.}}}$$

\mathcal{H}_S is span by the so-called Triplet:

$$\begin{cases} |+, +\rangle \\ \frac{1}{\sqrt{2}} (|+, -\rangle + |- , +\rangle) \\ |-, -\rangle \end{cases}$$

\mathcal{H}_A is span by the singlet: $\frac{1}{\sqrt{2}} (|+, -\rangle - |- , +\rangle) \equiv |T_A\rangle$

If $\mathbf{r}_1, \mathbf{r}_2$ are the coordinates of the two electrons, the overall state $|T\rangle$ of the two electrons must be totally antisymmetric, with respect to the exchange of the two particles. This means that either:

$$\langle \mathbf{r}_1, \mathbf{r}_2 | T \rangle = \phi_A(\mathbf{r}_1, \mathbf{r}_2) \begin{cases} |+, +\rangle \\ \frac{1}{\sqrt{2}} (|+, -\rangle + |- , +\rangle) \\ |-, -\rangle \end{cases}$$

$$\phi_A(\mathbf{r}_1, \mathbf{r}_2) = -\phi_A(\mathbf{r}_2, \mathbf{r}_1)$$

or

$$\langle \mathbf{r}_1, \mathbf{r}_2 | T \rangle = \phi_S(\mathbf{r}_1, \mathbf{r}_2) \frac{1}{\sqrt{2}} (|+, -\rangle - |- , +\rangle)$$

$$\phi_S(\mathbf{r}_1, \mathbf{r}_2) = \phi_S(\mathbf{r}_2, \mathbf{r}_1)$$

Any state of the form

$$| \psi \rangle = | \psi_1 \rangle \otimes | \psi_2 \rangle \\ = (c_+ | + \rangle + c_- | - \rangle) \otimes (d_+ | + \rangle + d_- | - \rangle)$$

is said to be SEPARABLE, or FACTORABLE.

Vice versa, a state of the form

$$| \psi_E \rangle = \sum_{i,j=1}^2 c_{ij} | i, j \rangle \quad \begin{matrix} 1 \leftrightarrow + \\ 2 \leftrightarrow - \end{matrix}$$

with $c_{ij} \neq a_i b_j$

is NON-separable or ENTANGLED.

This kind of state can always be written in a Schmidt form like:

$$| \psi_E \rangle = \sum_{i=1}^2 \sqrt{\lambda_i} | u_i \rangle | v_i \rangle$$

where $\langle u_i | u_j \rangle = \delta_{ij} = \langle v_i | v_j \rangle$ and $\lambda_i \geq 0$

Proof: Consider the coefficients c_{ij} as the elements of a matrix

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Then, make a Singular Value Decomposition of C :

$$C = U D V^T$$

where U, V are unitary matrices and D is diagonal:

$$D_{ij} = \delta_{ij} \sqrt{\lambda_j}$$

Therefore,

$$C_{ij} = \sum_{\kappa, \ell=1}^2 U_{i\kappa} D_{\kappa\ell} V_{\ell j}^+ \\ = \sum_{\kappa=1}^2 \sqrt{\lambda_\kappa} U_{i\kappa} V_{j\kappa}^* \Rightarrow$$

$$\Rightarrow |\psi_E\rangle = \sum_{ij} \left(\sum_{\kappa} \sqrt{\lambda_\kappa} U_{i\kappa} V_{j\kappa}^* \right) |i\rangle |j\rangle \\ = \sum_{\kappa} \sqrt{\lambda_\kappa} \underbrace{\left(\sum_i U_{i\kappa} |i\rangle \right)}_{\equiv |u_\kappa\rangle} \underbrace{\left(\sum_j V_{j\kappa}^* |j\rangle \right)}_{\equiv |v_\kappa\rangle} \\ = \sum_{\kappa} \sqrt{\lambda_\kappa} |u_\kappa, v_\kappa\rangle$$

By def:

$$\langle u_i | u_j \rangle = \sum_{\kappa, \ell} \underbrace{\langle \kappa | U_{\kappa i}^* \rangle \langle U_{\ell j} \rangle}_{\delta_{\kappa\ell}} \\ = \sum_{\kappa} U_{\kappa i}^* U_{\kappa j} \\ = \sum_{\kappa} U_{i\kappa}^+ U_{\kappa j} \\ = (U^+ U)_{ij} = \delta_{ij} \quad \text{c.v.d. (some proof for } \langle v_i | v_j \rangle)$$

This is called a Schmidt decomposition.

The generalization to N dimension is Trivial.

It is valid only for pure states.

The quantity

$$K = \frac{1}{\sum_{i=1}^2 \lambda_i^2}$$

is called the Schmidt rank of the state and gives a measure of entanglement. For a factorable state as

$$|+, +\rangle = |1, 1\rangle$$

$$\text{we have } \lambda_1 = 1, \lambda_2 = 0 \Rightarrow K = 1$$

For a singlet state:

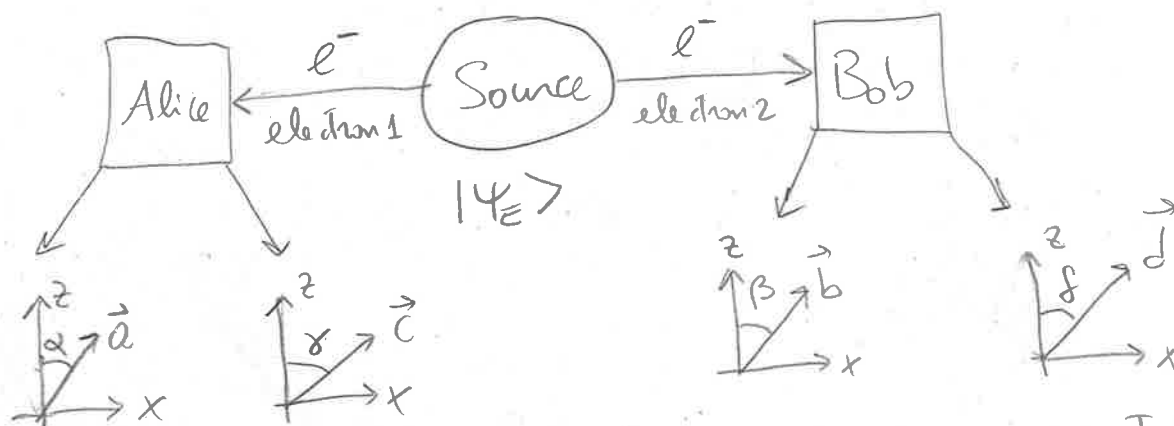
$$\frac{1}{\sqrt{2}}(|+, -\rangle - |- , +\rangle) = \frac{1}{\sqrt{2}}|u_1\rangle|v_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle|v_2\rangle$$

$$\text{where } \{|u_1\rangle, |u_2\rangle\} = \{|+\rangle, |-\rangle\}$$

$$\{|v_1\rangle, |v_2\rangle\} = \{|+\rangle, |-\rangle\}$$

$$\text{and } \sqrt{\lambda_1} = \sqrt{\lambda_2} = \frac{1}{\sqrt{2}} \Rightarrow K = 2 \text{ (Maximally entangled)}$$

- Bell's inequality -



Alice and Bob have two Stern-Gerlach (SG) setups. They can choose the orientation of the magnetic field parallel to either \vec{a} or \vec{c} (Alice) and parallel to either \vec{b} or \vec{d} (Bob).

The source emits pairs of electrons in the entangled state

$$|\Psi_E\rangle = \frac{1}{\sqrt{2}} (|+, -\rangle - |- , +\rangle)$$

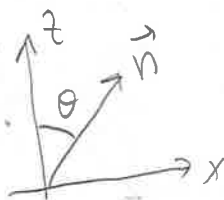
(z is reference quantization axis for the source)

From now on we choose the units:

$$\hbar = 1$$

and we recall that if we define $\hat{S}_{\vec{n}} \equiv \hat{S} \cdot \vec{n}$

with \vec{n} such that:



$$\vec{n} = \sin\theta \vec{e}_x + \cos\theta \vec{e}_z$$

Then

$$\hat{S}_{\vec{n}} |\vec{n}, \pm\rangle = \pm \frac{1}{2} |\vec{n}, \pm\rangle$$

where

$$\begin{cases} |\vec{n}, +\rangle = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} |-\rangle \\ |\vec{n}, -\rangle = -\sin \frac{\theta}{2} |+\rangle + \cos \frac{\theta}{2} |-\rangle \end{cases}$$

Let $P_+(\alpha)$ the probability that Alice detect an electron in the state $|\vec{a}, +\rangle$ irrespective of Bob's measurement is calculated as, for example,

$$P_+(\alpha) = |(\langle \vec{a}, + | + \rangle)^+ \langle \psi_E \rangle|^2 + |(\langle \vec{a}, + | - \rangle)^+ \langle \psi_E \rangle|^2$$

$$= |\langle \vec{a}, +; + | \psi_E \rangle|^2 + |\langle \vec{a}, +; - | \psi_E \rangle|^2$$

where

$$\langle \psi_E | \vec{a}, +; - \rangle = \frac{1}{\sqrt{2}} (\langle +, - | - \langle -, + |) \left(\cos \frac{\alpha}{2} |+\rangle + \sin \frac{\alpha}{2} |-\rangle \right) |-\rangle$$

$\begin{matrix} & & & = 0 \\ & & & \uparrow \\ & & = 1 & \\ & & & \downarrow \\ & & & = 0 \end{matrix}$

$$= \frac{1}{\sqrt{2}} \cos \frac{\alpha}{2}$$

$$\langle \psi_E | \vec{a}, +; + \rangle = -\frac{1}{\sqrt{2}} \sin \frac{\alpha}{2}$$

and $P_+(\alpha) = \frac{1}{2} \cos^2 \frac{\alpha}{2} + \frac{1}{2} \sin^2 \frac{\alpha}{2} = \frac{1}{2}$

The joint probability $P_+(\alpha, \beta)$ to detect the electrons in the state $|\vec{a}, +\rangle |\vec{b}, +\rangle$ is:

$$P_+(\alpha, \beta) = \left| \langle \vec{a}, + | \langle \vec{b}, + | \frac{1}{\sqrt{2}} (|+, -\rangle - |-, +\rangle) \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \left(\cos \frac{\alpha}{2} \sin \frac{\beta}{2} - \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \right) \right|^2 = \frac{1}{2} \sin^2 \left(\frac{\alpha - \beta}{2} \right)$$

Similarly, the probability

$$P_{+-}(\alpha, \beta)$$

to find the electrons in the state

$$|\vec{a}, +\rangle |\vec{b}, -\rangle$$

is

$$\begin{aligned}
P_{+-}(\alpha, \beta) &= \frac{1}{2} \left| \langle \vec{a}, + | + \rangle \langle \vec{b}, - | - \rangle - \langle \vec{a}, + | - \rangle \langle \vec{b}, - | + \rangle \right|^2 \\
&= \frac{1}{2} \left| \cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \left(-\sin \frac{\beta}{2} \right) \right|^2 \\
&= \frac{1}{2} \left| \cos \left(\frac{\alpha - \beta}{2} \right) \right|^2 = \frac{1}{2} \cos^2 \left(\frac{\alpha - \beta}{2} \right)
\end{aligned}$$

Finally, the conditional probability

$$P_c(\alpha | \beta)$$

that Alice finds her electrons either in $|\vec{a}, +\rangle$ or $|\vec{a}, -\rangle$ given that Bob has detected his electron in either $|\vec{b}, +\rangle$ or $|\vec{b}, -\rangle$, is equal to:

$$\begin{aligned}
P(\vec{a}, +; \vec{b}, +) + P(\vec{a}, -; \vec{b}, +) &= P(\vec{a}, +; \vec{b}, -) + P(\vec{a}, +; \vec{b}, -) \\
&= \sin^2 \left(\frac{\alpha - \beta}{2} \right)
\end{aligned}$$

The outcome of the measurement on electron 1 appears to be influenced by the orientation β of Bob's setup, even though the two electrons may be well separated at the time of measurement. This is the so-called non-locality paradox.