Exercises 04.05.2017

Problem 1

Consider the linear vector space over the field $\mathbb{C}$ whose elements are the polynomials $x(t) \in \mathbb{C}$,

$$x(t) = c_0 + c_1 t + \ldots + c_n t^n, \qquad (n = 0, 1, 2, \ldots). \tag{1}$$

What of the following sets are linear vector spaces?

(a) Polynomials of degree 3, $x_3(t)$.

(b) Polynomials of degree $n \leq 3$, $x_n(t) : n \leq 3$.

(c) Polynomials such that $2x(0) = x(1)$.

(d) Polynomials such that $3x(0) = x(1) + c$, with $c \in \mathbb{C}$.

(e) Polynomials such that $x(t) \geq 0$ for $0 \leq t \leq 1$.

Solution 1

(a) This is not a vector space because the sum of two polynomials of degrees 3 is a polynomial of degree $n \leq 3$. For example, given

$$x_3(t) = 1 + 2t - 3t^3, \qquad y_3(t) = 16 + 7t^2 + 3t^3, \tag{2}$$

then

$$x_3(t) + y_3(t) = 17 + 2t + 7t^2, \tag{3}$$

which is a polynomial of second degree.

(b) This is a vector space, because the sum of two polynomials of degrees $n \leq 3$ is still a polynomial of degree $n \leq 3$.

(c) This is a vector space, because if $x(t) : 2x(0) = x(1)$ and $y(t) : 2y(0) = y(1)$, then $z(t) = ax(t) + by(t)$ is such that $2z(0) = z(1)$. The other properties are easily verified.

(d) This is not a vector space, because if $x(t) : 3x(0) = x(1) + c$ and $y(t) : 3y(0) = y(1) + c$, then $z(t) = ax(t) + by(t)$ is such that $3z(0) = z(1) + c(a + b) \neq z(1) + c$ (except for the very peculiar case $a + b = 1$).

(e) This is not a vector space, because given any $x(t) \geq 0$, the negative element $(-1)x(t) = -x(t) \leq 0$ does not belong to the space of nonnegative polynomials.

Problem 2

For what values of the variable x the vectors

$$|\psi_1\rangle = (1, 0, x), \qquad |\psi_2\rangle = (0, x, 1), \qquad |\psi_3\rangle = (1 + x, x, 1), \tag{4}$$

form a basis for $\mathbb{C}^3$?

Solution 2

By definition, a basis of $\mathbb{C}^3$ is any set of three linearly independent vectors. The three vectors $\{|\psi_i\rangle\}_{i=1}^3$ are linearly independent if it is not possible to find three numbers a, b and c, not all zero, such that

$$a|\psi_1\rangle + b|\psi_2\rangle + c|\psi_3\rangle = 0. \tag{5}$$

Putting equal to zero the three components of the vector $a|\psi_1\rangle + b|\psi_2\rangle + c|\psi_3\rangle$, we obtain an algebraic system of equations that may be written as,

$$M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6}$$

where the 3×3 matrix M is defined as:

$$M \equiv \begin{bmatrix} 1 & 0 & 1+x \\ 0 & x & x \\ x & 1 & 1 \end{bmatrix} \tag{7}$$

The system (6) does not possess solutions if $\det M \neq 0$, namely if

$$\det M = -x^2(1 + x) \neq 0. \tag{8}$$

Therefore, we must have

$$x \neq 0 \qquad \text{and} \qquad x \neq -1, \tag{9}$$

so that $|\psi_1\rangle, |\psi_2\rangle$ and $|\psi_3\rangle$ form a basis of $\mathbb{C}^3$.

Problem 3

Consider the linear vector space over the field $\mathbb{C}$ whose elements are the polynomials

$$x(t) = c_0 + c_1 t + c_2 t^2, \tag{10}$$

of degree less or equal to 2, and consider the operator $\hat{A}$ that maps $x(t)$ into $x(1+t)$, namely

$$x \to y = \hat{A}x \qquad \text{such that} \qquad (\hat{A}x)(t) = x(1+t). \tag{11}$$

Calculate the matrix $A = [a_{ij}]$ representing the operator $\hat{A}$ in the basis

$$|e_0\rangle = 1, \qquad |e_1\rangle = t, \qquad |e_2\rangle = t^2, \tag{12}$$

without using the notion of scalar product. Then, using this representation, calculate $y(t) = (\hat{A}x)(t)$ for $x(t) = 2 + 3t + t^2$.

Solution 3

By definition, given $\hat{A}|x\rangle = |y\rangle$, with

$$|x\rangle = \sum_{j=0}^{2} c_j |e_j\rangle, \qquad \text{and} \qquad |y\rangle = \sum_{i=0}^{2} d_i |e_i\rangle, \tag{13}$$

the matrix elements of the matrix $A = [a_{ij}]$ representing the operator $\hat{A}$, are defined by the relation

$$d_i = \sum_{j=0}^{2} a_{ij} c_j. \tag{14}$$

From the definition (11) it follows that

$$
\begin{aligned}
(\hat{A}x)(t) &= x(1+t) \\
&= c_0 + c_1(1+t) + c_2(1+t)^2 \\
&= (c_0 + c_1 + c_2) + (c_1 + 2c_2)t + c_2 t^2 \\
&\equiv d_0 + d_1 t + d_2 t^2. \qquad (15)
\end{aligned}
$$

The equality of the last two lines of the equation above can be written as:

$$
\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}, \qquad (16)
$$

where the 3×3 matrix A representing the operator $\hat{A}$ in the basis $\{|e_i\rangle\}_{i=0}^{2}$, is given by

$$
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (17)
$$

Finally, to answer the second question, if $x(t) = 2 + 3t + t^2$, then $y(t) = (\hat{A}x)(t) = d_0 + d_1 t + d_2 t^2$ can be calculated either via a matrix multiplication,

$$
\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \qquad (18)
$$

or directly as

$$
\begin{aligned}
(\hat{A}x)(t) &= x(1+t) \\
&= 2 + 3(1+t) + (1+t)^2 \\
&= (2+3+1) + (3+2)t + t^2 \\
&\equiv 6 + 5t + t^2. \qquad (19)
\end{aligned}
$$

Problem 4

Consider the vector $|x\rangle = (x_1, x_2, x_3) \in \mathbb{C}^3$ and the three operators $\hat{S}_1, \hat{S}_2$ and $\hat{S}_3$

defined via

$$\hat{S}_1|x\rangle = (x_1, x_3, x_2), \tag{20}$$
$$\hat{S}_2|x\rangle = (x_3, x_2, x_1), \tag{21}$$
$$\hat{S}_3|x\rangle = (x_2, x_1, x_3). \tag{22}$$

Determine the corresponding matrices S_1, S_2 and S_3 with respect to the standard basis

$$|e_1\rangle = (1, 0, 0), \qquad |e_2\rangle = (0, 1, 0), \qquad |e_3\rangle = (0, 0, 1), \tag{23}$$

and show that these matrices are unitary.

Solution 4

From (20) and the definition

$$\langle e_i|\hat{A}|v\rangle = \sum_{j=1}^{3} a_{ij}\langle e_j|v\rangle, \qquad (i = 1, 2, 3), \tag{24}$$

valid for any linear operator $\hat{A}$ and vector $|v\rangle \in \mathbb{C}^3$, we can calculate straightforwardly

$$\begin{aligned}
\langle e_1|\hat{S}_1|x\rangle &= [S_1]_{11}x_1 + [S_1]_{12}x_2 + [S_1]_{13}x_3 = x_1, \\
\langle e_2|\hat{S}_1|x\rangle &= [S_1]_{21}x_1 + [S_1]_{22}x_2 + [S_1]_{23}x_3 = x_3, \\
\langle e_3|\hat{S}_1|x\rangle &= [S_1]_{31}x_1 + [S_1]_{32}x_2 + [S_1]_{33}x_3 = x_2.
\end{aligned} \tag{25}$$

These relations imply that

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{26}$$

In a similar way we can calculate

$$S_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad S_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{27}$$

All these matrices are real-valued and symmetric and, therefore, $S_i S_i^\dagger = S_i^\dagger S_i = S_i^2$. To demonstrate the unitary nature of these matrices we have to show that

$$S_1 S_1^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{28}$$

and so on for the other matrices.

Problem 5

In the lectures we have seen that the most general solution of the one-dimensional Schrödinger equation for a free particle of mass m,

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t), \tag{29}$$

can be found using the Fourier transform theorem with respect to the variable x to write,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k,t) e^{ikx}\, dk, \tag{30}$$

where

$$\phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{-ikx}\, dx. \tag{31}$$

Inserting (30) in the Schrödinger equation, we obtain an ordinary differential equation for $\phi(k,t)$,

$$i\hbar \frac{d}{dt}\phi(k,t) = \frac{\hbar^2 k^2}{2m}\phi(k,t), \tag{32}$$

whose solution is,

$$\phi(k,t) = \exp\left(-i\frac{\hbar^2 k^2}{2m}\frac{t}{\hbar}\right)\phi(k,0). \tag{33}$$

Substituting this equation into (30) we obtain

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k,0) \exp\left[-i\left(\frac{\hbar^2 k^2}{2m}\frac{t}{\hbar} - kx\right)\right]\, dk. \tag{34}$$

It should be noticed that the physical dimensions $[\hbar k]$ are those of a linear momentum. This suggests that it may be convenient to change the integration variable from k to $p = \hbar k$ in the integral above, and to rewrite it as

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(p) \exp\left[-i\left(\frac{p^2}{2m}\frac{t}{\hbar} - \frac{px}{\hbar}\right)\right] dp, \tag{35}$$

where we have defined

$$\varphi(p) \equiv \frac{1}{\sqrt{\hbar}}\, \phi(p/\hbar, 0). \tag{36}$$

The normalization condition requires that

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\, dx \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left\{ \left[\int_{-\infty}^{\infty} \varphi^*(p) \exp\left[i\left(\frac{p^2}{2m}\frac{t}{\hbar} - \frac{px}{\hbar}\right)\right] dp \right] \left[\int_{-\infty}^{\infty} \varphi(p') \exp\left[-i\left(\frac{p'^2}{2m}\frac{t}{\hbar} - \frac{p'x}{\hbar}\right)\right] dp' \right] \right\} dx \\
&= \iint_{-\infty}^{\infty} \left\{ \varphi^*(p)\varphi(p') \exp\left[i\frac{t}{\hbar}\frac{1}{2m}\left(p^2 - p'^2\right)\right] \underbrace{\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \exp\left[-ix\left(\frac{p-p'}{\hbar}\right)\right] dx}_{=\,\delta\,(p-p')} \right\} dp\, dp' \\
&= \int_{-\infty}^{\infty} |\varphi(p)|^2\, dp.
\end{aligned} \tag{37}$$

Note that this equality is a straightforward consequence of the Parseval's theorem that we have previously studied. Now, we *arbitrarily* choose

$$\varphi(p) = N^{1/2} \exp\left[-\frac{(p-p_0)^2}{\Delta p^2}\right], \tag{38}$$

where p_0 is a real number, Δp is a positive number and N is the so-called *normalization* *constant*, which is fixed by imposing the normalization constraint

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\varphi(p)|^2 \, dp \\
&= N \int_{-\infty}^{\infty} \exp\left[-2\frac{(p-p_0)^2}{\Delta p^2}\right] dp \\
&= N \Delta p \left(\frac{\pi}{2}\right)^{1/2}.
\end{aligned} \tag{39}
$$

Therefore, we take $N = \sqrt{2/\pi}/\Delta p$, and we calculate the integral (35) with $\varphi(p)$ given by (38). The result is

$$
\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2}{\pi\Delta p^2}\right)^{1/4} \int_{-\infty}^{\infty} \exp\left[-\frac{(p-p_0)^2}{\Delta p^2} - i\left(\frac{p^2}{2m}\frac{t}{\hbar} - \frac{px}{\hbar}\right)\right] dp \\
&= \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{\frac{2\hbar}{\Delta p}\left(1 + i\,t\,\frac{\Delta p^2}{2m\hbar}\right)}} \exp\left[-\frac{x^2}{4\left(\frac{\hbar}{\Delta p}\right)^2\left(1 + i\,t\,\frac{\Delta p^2}{2m\hbar}\right)} - \frac{i}{\hbar}\frac{\left(p_0\,x - \frac{p_0^2}{2m}\,t\right)}{1 + i\,t\,\frac{\Delta p^2}{2m\hbar}}\right],
\end{aligned} \tag{40}
$$

where we have used the following formula of Gaussian integration:

$$
\int_{-\infty}^{\infty} \exp\left(-\alpha x^2 + \beta x\right) dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right), \qquad \alpha, \beta \in \mathbb{C}, \quad \mathrm{Re}\,\alpha > 0. \tag{41}
$$

If we define

$$
\Delta x \equiv \frac{\hbar}{\Delta p}, \qquad t_0 \equiv \frac{2m\hbar}{\Delta p^2} = \frac{2m\,\Delta x^2}{\hbar}, \qquad \hbar\omega_0 \equiv \frac{p_0^2}{2m}, \qquad k_0 \equiv \frac{p_0}{\hbar}, \tag{42}
$$

we can rewrite $\psi(x,t)$ as

$$
\psi(x,t) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\Delta x\left(1 + i\,\frac{t}{t_0}\right)}} \exp\left[-\frac{x^2}{4\Delta x^2\left(1 + i\,\frac{t}{t_0}\right)} - i\,\frac{k_0 x - \omega_0 t}{1 + i\,\frac{t}{t_0}}\right]. \tag{43}
$$

The corresponding probability density is

$$P(x,t) = |\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\,\sigma(t)} \exp\left[-\frac{(x-v_0t)^2}{2\sigma^2(t)}\right], \tag{44}$$

where we have defined the velocity $v_0 \equiv p_0/m$ and the time-dependent variance

$$\sigma^2(t) \equiv \Delta x^2 \left(1 + t^2/t_0^2\right). \tag{45}$$

This probability density moves with velocity v_0 and spreads according to (45). This can be seen calculating the first two moments of $P(x,t)$, the mean

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x P(x,t)\, dx \\ &= v_0 t, \end{aligned} \tag{46}$$

and the variance

$$\begin{aligned} \langle (x - \langle x \rangle)^2 \rangle &= \int_{-\infty}^{\infty} (x - v_0 t)^2\, P(x,t)\, dx \\ &= \sigma^2(t). \end{aligned} \tag{47}$$

At time $t = t_0$ the probability density is wider by a factor $\sqrt{2}$:

$$\sigma(t_0) = \sqrt{2}\,\Delta x. \tag{48}$$

For an object of mass $m = 1\,\mathrm{g}$ and size $\Delta x = 1\,\mathrm{cm}$, the "spreading time" t_0 is about

$$t_0 \simeq 2 \times 10^{27}\,\mathrm{s} \simeq 6 \times 10^{19}\,\mathrm{yr}. \tag{49}$$

Therefore, if the object moves at velocity $v_0 = 1\,\mathrm{m/s}$, in the time t_0 it will have traveled by a distance $v_0 t_0 \simeq 2 \times 10^{27}\,\mathrm{m}$. Considering that the diameter of the observable universe is $\sim 10^{27}\,\mathrm{m}$ and its age is $\sim 10^{10}\,\mathrm{yr}$, it is clear why macroscopic objects do not experience any sensible quantum mechanical spreading. It is interesting to see how quickly this Gaussian probability density spreads. To this end, we can calculate

$$\frac{d\sigma}{dt} = \frac{\Delta x\, t}{t_0^2 \sqrt{1 + t^2/t_0^2}} \qquad \text{and} \qquad \lim_{t\to\infty} \frac{d\sigma}{dt} = \frac{\Delta x}{t_0} = \frac{\hbar}{2m\Delta x}. \tag{50}$$

For our specific example, we obtain a very small limiting velocity:

$$\frac{\Delta x}{t_0} \simeq 5 \times 10^{-30}\,\mathrm{m/s}. \tag{51}$$