Friedrich-Alexander Universität Erlangen-Nürnberg

WS 2012/2013

Institut für Theoretische Physik II
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Sprechstunde: Do. 11-12 Uhr, Raum 02.782.
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12. Übungsblatt Many-body physics with ultra-cold atomic gases

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12.1: BCS theory, variational approach, questions

Consider the BCS Hamiltonian:

$$K = H - \mu N = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{V_0}{V} \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} \tag{1}$$

with $\xi_k = \epsilon_k - \mu$ and $V_0 < 0$. V is the volume. The variational approach uses this ansatz:

$$|BCS\rangle = \prod_k (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle \tag{2}$$

where u_k and v_k are real numbers ($u_k = u_{-k}, v_k = v_{-k}$).

a. Show that $|BCS\rangle$ is normalized if

$$u_k^2 + v_k^2 = 1\,. \tag{3}$$

b. Compute $\langle BCS|N|BCS\rangle$ where $N = \sum_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma}$.

c. Compute $\langle BCS|c_{-k\downarrow} c_{k\uparrow}|BCS\rangle$.

d. Compute the expectation value of K

$$E_{\rm BCS} = \langle BCS|K|BCS\rangle$$

and express it in terms of the variational parameters u_k and v_k. Use the ansatz $u_k = \sin(\Theta_k)$, $v_k = \cos(\Theta_k)$ and minimize the $E_{\rm BCS}$ with respect to Θ_k. This leads to a condition for Θ_k, in which the gap function

$$\Delta = -\frac{V_0}{2V} \sum_k \sin 2\Theta_k$$

should appear.

e. How can the theory be extended to finite temperatures? Which form does the gap equation take then?

f. Which equation determines the unknown chemical potential μ?

g. Argue that the BCS wave-function is a coherent state.

h. Plot u_k and v_k for (i) $V_0 = 0$ and (ii) $V_0 < 0$.

i. From the result for the gap parameter derived in the lecture, compare 1D and 3D. Argue that pairing is more robust at low densities in 1D, while in 3D the opposite is correct.

General questions: Bose gases

(a) Explain the derivation of the time-independent Gross-Pitaevskii equation.

(b) What is the Bogoliubov approximation for a weakly interacting Bose gas?

(c) What is a Tonks-Girardeau gas? Explain the term fermionization of a one-dimensional Bose gas.

(d) Which types of excitations exist in a weakly interatcing Bose gas? Which ones are responosible for destroying off-diagonal long-range order in the one-body density matrix? Which ones are relevant for the BKT transition in 2D?

(e) What's the basic ansatz in any mean-field approach? What are known short-comings of mean-field theory?

(f) For weakly interacting bosons, what is the Thomas-Fermi approximation? What form does the Gross-Pitaevskii equation take and what is the solution?

(g) What is Landau's criterion for superfluidity? Sketch the dispersion of excitations above the BEC state and graphically determine the critical velocity.

(h) What is definition of the superfluid density? What is the definition of the condensate fraction?

(i) What is a symmetry-broken phase? What is long-range order?

(j) What's the condition that determines the critical temperature of an ideal Bose gas?

(k) Discuss the qualitative behaviour of the condensate fraction of an ideal Bose gas in a box in 3D as a function of temperature. What's the behaviour close to T_c?