Friedrich-Alexander Universität Erlangen-Nürnberg

WS 2012/2013

Institut für Theoretische Physik II
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1. Übungsblatt Many-body physics with ultra-cold atomic gases

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1.1: Identical particles

We consider two identical, non-interacting particles with mass m and assume that the Hamiltonian is

$$H = \sum_{i=1}^{2} H_i$$

with the one-particle Hamiltonian

$$H_i = \frac{\boldsymbol{p}_i^2}{2m} + V(\boldsymbol{r}_i)\,.$$

We denote the one-particle eigenfunction as $\varphi_n(\boldsymbol{r}_i)$ with one-particle energies E_n and $V(\boldsymbol{r}_i)$ ia a one-body potential. Write down the ground-state energy and wave function for the two particles in terms of E_n and φ_n, respectively, for these cases:

a. Two spinless bosons.

b. Two spinless fermions.

c. Two fermions with spin 1/2.

1.2: Partition function, Bose and Fermi function

We consider a free gas of N identical particles and assume that all single-particle states $|\psi_r\rangle$ and energies ϵ_r are known.

(a) The partition function in the canonical ensemble is

$$Z = tr(e^{-\beta H})\,,$$

where $\beta = 1/(k_B T)$ is the inverse temperature. Show that the total energy E can be written as:

$$E = -\frac{\partial lnZ}{\partial \beta}\,.$$

(b) We next consider the grand-canonical ensemble. Show that the partition function can be written as

$$Z = tr(e^{-\beta(H-\mu\hat{N})}) = \begin{cases} \prod_r (1 - e^{-\beta(\epsilon_r-\mu)})^{-1} & \text{for} \quad \text{bosons} \\ \prod_r (1 + e^{-\beta(\epsilon_r-\mu)}) & \text{for} \quad \text{fermions} \end{cases}$$

$\hat{N}$ is the operator whose eigenvalue in a many-body state with fixed N is N, μ is the chemical potential. Use the fact that total energy E and total particle number N can be written as

$$E = \sum_r n_r \epsilon_r \, ; \quad N = \sum_r n_r$$

using the occupation numbers n_r that were introduced in the lecture.

(c) Next, compute the mean occupation $\hat{n}_r$ in the single-particle eigenstate with eigenenergy ϵ_r and show that this leads to the Bose- and Fermi-distribution functions:

$$\langle \hat{n}_r \rangle = \begin{cases} \frac{1}{\exp(\beta(\epsilon_r-\mu)-1)} & \text{for} \quad \text{bosons} \\ \frac{1}{\exp(\beta(\epsilon_r-\mu)+1)} & \text{for} \quad \text{fermions} \end{cases} .$$

1.3: Ideal Bose gas in 2D

Argue that T_c for an ideal Bose gas confinded into a box of area $A = L^2$ is zero.

1.4: de-Broglie-Wavelength

(a) At which temperature does the thermal de-Broglie-wavelength $\lambda_{\text{dB}} = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$ of a Rubidium atom (^{87}Rb, $m = 87u$) coincide with the wavelength of an Nd:YAG laser with a wavelength of $\lambda = 1064\,\text{nm}$? What is a typical velocity at this temperature? (The kinetic energy of an atom is given by $E_{\text{kin}} = 3/2\,k_BT$.)

(b) At what critical density n_c (typical units: cm^{-3}) would atoms with the above de-Broglie-wavelength start to condense if you assume the approximate relationship $n_c \cdot \lambda_{dB}^3 \approx 1$?