Lecture 14

Angular momentum operator algebra

In this lecture we present the theory of angular momentum operator algebra in quantum mechanics.

14.1 Basic relations

Consider the three Hermitian angular momentum operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$, which satisfy the commutation relations

$$
\left[\hat{J}_x, \hat{J}_y\right] = i\hbar \hat{J}_z, \quad \left[\hat{J}_y, \hat{J}_z\right] = i\hbar \hat{J}_x, \quad \left[\hat{J}_z, \hat{J}_x\right] = i\hbar \hat{J}_y.
$$

(14.1)

The operator

$$
\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2,
$$

(14.2)

is also Hermitian and it commutes with $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$:

$$
\left[\hat{J}^2, \hat{J}_x\right] = \left[\hat{J}^2, \hat{J}_y\right] = \left[\hat{J}^2, \hat{J}_z\right] = 0.
$$

(14.3)

These relations are not difficult to prove using the operator identity

$$
\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right].
$$

(14.4)
For example,

\[
[\hat{J}^2, \hat{J}_z] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_z] \\
= [\hat{J}_x^2, \hat{J}_z] + [\hat{J}_y^2, \hat{J}_z],
\]

(14.5)
because by definition \(\hat{J}_z\) commutes with itself and, using (14.4),

\[
[\hat{J}^2, \hat{J}_z] = -[\hat{J}_z, \hat{J}_z] \hat{J}_z - \hat{J}_z[\hat{J}_z, \hat{J}_z] = 0.
\]

(14.6)
The remaining two commutators in the last row of (14.5) can be calculated using again (14.4):

\[
[\hat{J}_x^2, \hat{J}_z] = [\hat{J}_x, \hat{J}_z] \hat{J}_x + \hat{J}_x [\hat{J}_x, \hat{J}_z] \\
= -i\hbar \left( \hat{J}_y \hat{J}_x + \hat{J}_x \hat{J}_y \right),
\]

(14.7)
and

\[
[\hat{J}_y^2, \hat{J}_z] = [\hat{J}_y, \hat{J}_z] \hat{J}_y + \hat{J}_y [\hat{J}_y, \hat{J}_z] \\
= i\hbar \left( \hat{J}_x \hat{J}_y + \hat{J}_y \hat{J}_x \right).
\]

(14.8)
The sum of \([\hat{J}_x^2, \hat{J}_z]\) and \([\hat{J}_y^2, \hat{J}_z]\) is therefore zero and from (14.5) it follows that \([\hat{J}^2, \hat{J}_z] = 0\).

Rather then working with the Hermitian operators \(\hat{J}_x\) and \(\hat{J}_y\), it is more convenient to work with the non-Hermitian linear combinations,

\[
\hat{J}_+ = \hat{J}_x + i\hat{J}_y, \\
\hat{J}_- = \hat{J}_x - i\hat{J}_y,
\]

(14.9a)

(14.9b)
where, by definition, \((\hat{J}_-)\dagger = \hat{J}_+\). For reasons that will become clear later, \(\hat{J}_+\) and \(\hat{J}_-\) are called ladder operators. Using (14.1) and (14.3), it is straight-
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forward to show that

\[
\begin{align*}
\hat{J}_z \hat{J}_+ &= \hbar \hat{J}_+, \\
\hat{J}_z \hat{J}_- &= -\hbar \hat{J}_-, \\
\hat{J}_+ \hat{J}_- &= 2\hbar \hat{J}_z, \\
\hat{j}^2, \hat{J}_+ &= [\hat{j}^2, \hat{J}_+] = 0.
\end{align*}
\] (14.10a)

The operators \(\hat{J}_+\) and \(\hat{J}_-\) often appear in the products \(\hat{J}_+ \hat{J}_-\) and \(\hat{J}_- \hat{J}_+\), which are equal to

\[
\hat{J}_+ \hat{J}_- = \left( \hat{j}_x + i \hat{j}_y \right) \left( \hat{j}_x - i \hat{j}_y \right) = \hat{j}_x^2 + \hat{j}_y^2 - i [\hat{j}_x, \hat{j}_y] = \hat{j}_x^2 + \hat{j}_y^2 + \hbar \hat{J}_z,
\] (14.11)

and

\[
\hat{J}_- \hat{J}_+ = \left( \hat{j}_x - i \hat{j}_y \right) \left( \hat{j}_x + i \hat{j}_y \right) = \hat{j}_x^2 + \hat{j}_y^2 + i [\hat{j}_x, \hat{j}_y] = \hat{j}_x^2 + \hat{j}_y^2 - \hbar \hat{J}_z,
\] (14.12)

respectively. After noticing that (14.2) implies \(\hat{j}_x^2 + \hat{j}_y^2 = \hat{j}^2 - \hat{j}^2_z\), we can straightforwardly rewrite \(\hat{J}_+ \hat{J}_-\) and \(\hat{J}_- \hat{J}_+\) as

\[
\begin{align*}
\hat{J}_+ \hat{J}_- &= \hat{j}^2 - \hat{j}_z^2 + \hbar \hat{J}_z, \\
\hat{J}_- \hat{J}_+ &= \hat{j}^2 - \hat{j}_z^2 - \hbar \hat{J}_z.
\end{align*}
\] (14.13a)

Adding equations (14.13) side-by-side and rearranging the terms, we obtain

\[
\hat{j}^2 = \frac{1}{2} \left( \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ \right) + \hat{j}_z^2.
\] (14.14)
Two additional useful relations are:

\[
[\hat{J}_z, (\hat{J}_+)^n] = n\hbar (\hat{J}_+)^n, \tag{14.15a}
\]

\[
[\hat{J}_z, (\hat{J}_-)^n] = -n\hbar (\hat{J}_-)^n, \tag{14.15b}
\]

where \( n = 0, 1, 2, \ldots \), is an integer number. We demonstrate, for example, the first relation (14.15a). The proof is by iteration. If \( n = 0 \), then \( (\hat{J}_+)^0 = \hat{I} \) and (14.15a) becomes a trivial identity. If \( n = 1 \), we simply recover (14.10a). If \( n > 1 \), we first rewrite

\[
\hat{J}_z (\hat{J}_+)^n = \hat{J}_z \hat{J}_+ (\hat{J}_+)^{n-1},
\]

and then use (14.10a) to write \( \hat{J}_z \hat{J}_+ = \hat{J}_+ \hat{J}_z + \hbar \hat{J}_+ \). This permits us to find the following recursion relation:

\[
\hat{J}_z (\hat{J}_+)^n = (\hat{J}_+ \hat{J}_z + \hbar \hat{J}_z) (\hat{J}_+)^{n-1}
= \hbar (\hat{J}_+)^n + \hat{J}_+ \left[ \hat{J}_z (\hat{J}_+)^{n-1} \right]. \tag{14.17}
\]

If we rewrite this equation replacing \( n \) with \( n - 1 \), we easily find

\[
\hat{J}_z (\hat{J}_+)^{n-1} = \hbar (\hat{J}_+)^{n-1} + \hat{J}_+ \left[ \hat{J}_z (\hat{J}_+)^{n-2} \right]. \tag{14.18}
\]

Substituting this expression into (14.17), we obtain

\[
\hat{J}_z (\hat{J}_+)^n = \hbar (\hat{J}_+)^n + \hat{J}_+ \left\{ \hbar (\hat{J}_+)^{n-1} + \hat{J}_+ \left[ \hat{J}_z (\hat{J}_+)^{n-2} \right] \right\}
= 2\hbar (\hat{J}_+)^n + (\hat{J}_+)^2 \left[ \hat{J}_z (\hat{J}_+)^{n-2} \right]. \tag{14.19}
\]

This procedure can be iterated again and again. After \( p \) iterations we find

\[
\hat{J}_z (\hat{J}_+)^n = p\hbar (\hat{J}_+)^n + (\hat{J}_+)^p \left[ \hat{J}_z (\hat{J}_+)^{n-p} \right], \tag{14.20}
\]

which becomes, for \( p = n \),

\[
\hat{J}_z (\hat{J}_+)^n = n\hbar (\hat{J}_+)^n + (\hat{J}_+)^n \hat{J}_z. \tag{14.21}
\]
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This proves (14.15a). Equation (14.15b) can be obtained in the same way using (14.10b).

Finally, from \([\hat{J}^2, \hat{J}_\pm] = 0\) and the equation\(^1\)

\[
[\hat{A}, \hat{B}^n] = [\hat{A}, \hat{B}] \hat{B}^{n-1} + \hat{B} [\hat{A}, \hat{B}] \hat{B}^{n-2} + \ldots + \hat{B}^{n-1} [\hat{A}, \hat{B}],
\]

(14.22)

where \(n\) is a positive integer number, it follows that

\[
[\hat{J}^2, (\hat{J}_\pm)^n] = 0.
\]

(14.23)

14.2 Compatible and incompatible observables

In quantum mechanics, two observables \(A\) and \(B\) are said to be compatible when the corresponding operators \(\hat{A}\) and \(\hat{B}\) commute,

\[
[\hat{A}, \hat{B}] = 0,
\]

(14.24)

and incompatible when they do not

\[
[\hat{A}, \hat{B}] \neq 0.
\]

(14.25)

If \(A\) and \(B\) are compatible and \(|\psi\rangle\) is an eigenvector of \(\hat{A}\) associated with the eigenvalue \(a\),

\[
\hat{A}|\psi\rangle = a|\psi\rangle,
\]

(14.26)

then \(\hat{B}|\psi\rangle\) is also an eigenvector of \(\hat{A}\) associated with the same eigenvalue \(a\). This can be easily seen applying \(\hat{B}\) to both sides of (14.26):

\[
\hat{B}\hat{A}|\psi\rangle = a\hat{B}|\psi\rangle.
\]

(14.27)

Since, by hypothesis, \([\hat{A}, \hat{B}] = 0\), we can replace \(\hat{B}\hat{A}\) on the left side of this equation with \(\hat{A}\hat{B}\), thus obtaining

\[
\hat{A} \left( \hat{B}|\psi\rangle \right) = a \left( \hat{B}|\psi\rangle \right).
\]

(14.28)

Now, there are two possibilities:

\(^1\)This equation can be easily proved by induction using (14.4).
1) If $a$ is a non-degenerate eigenvalue, then all vectors $|\psi\rangle$ satisfying (14.26) are parallel\(^2\) and $\hat{B}|\psi\rangle$ is necessarily proportional to $|\psi\rangle$, that is

$$\hat{B}|\psi\rangle = b|\psi\rangle. \tag{14.29}$$

Therefore, $|\psi\rangle$ is also an eigenvector of $\hat{B}$.

2) If $a$ is a degenerate eigenvalue, then the set of all vectors $|\psi\rangle$ satisfying (14.26) spans a subspace $\mathcal{E}_a$ associated with the eigenvalue $a$. Then, if $|\psi\rangle \in \mathcal{E}_a$, all what we can say is that $\hat{B}|\psi\rangle$ belongs to $\mathcal{E}_a$:

$$\hat{B}|\psi\rangle \in \mathcal{E}_a. \tag{14.30}$$

In the first case, the knowledge of the common eigenvector $|\psi\rangle$ of $\hat{A}$ and $\hat{B}$, uniquely determines the values of both $a$ and $b$. This means that $a$ and $b$ are not independent variables and the knowledge of $a$ fixes the value of $b$ and vice versa. In the second case, instead, we need to specify both values $a$ and $b$ to uniquely identify the common eigenvectors of $\hat{A}$ and $\hat{B}$. A fundamental theorem of quantum mechanics states that\(^3\):

*When two observables $A$ and $B$ are compatible, it is always possible to find an orthonormal basis of the vector space spanned by the eigenvectors common to the corresponding operators $\hat{A}$ and $\hat{B}$."

We denote with $\{|a,b\rangle\}$ such a basis, where the eigenvectors $|a,b\rangle$ are defined by the properties

$$\hat{A}|a,b\rangle = a|a,b\rangle, \tag{14.31a}$$

$$\hat{B}|a,b\rangle = b|a,b\rangle. \tag{14.31b}$$

\(^2\)Here and hereafter, the statement that two vectors $|\psi\rangle$ and $|\phi\rangle$ are parallel, signifies that $|\psi\rangle = \gamma|\phi\rangle$, where $\gamma$ is any nonzero complex number.

and

\[ \langle a', b'|a, b\rangle = \delta_{aa'}\delta_{bb'}. \quad (14.32) \]

If the ordered pair of eigenvalues \((a, b)\) determine a unique common eigenvector \(|a, b\rangle\) of \(\hat{A}\) and \(\hat{B}\), then the set of observables \(\{A, B\}\) is called a Complete Set of Commuting Observables (CSCO), or a maximal set of commuting observables; that is there is not a third observable \(C\) such that

\[ [\hat{A}, \hat{B}] = [\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0. \quad (14.33) \]

The eigenvalues of operators \(\hat{A}\) and \(\hat{B}\) may still be degenerate, but if we specify a pair \((a, b)\), then the corresponding eigenvector \(|a, b\rangle\) common to \(\hat{A}\) and \(\hat{B}\) is uniquely specified.

The Hermitian operator \(\hat{A}\) possess at least one degenerate eigenvalue when there are two observables \(B\) and \(C\) compatible with \(A\) but incompatible each other.

To prove this statement, consider three observables \(A, B\) and \(C\) such that \(\{A, B\}\) is a CSCO, with

\[ [\hat{A}, \hat{B}] = 0 = [\hat{A}, \hat{C}], \quad \text{and} \quad [\hat{B}, \hat{C}] \neq 0. \quad (14.34) \]

According to our nomenclature, equations (14.34) imply that \(A\) is compatible with both \(B\) and \(C\), but \(B\) and \(C\) are incompatible. Let \(|a, b\rangle\) be a common eigenvector of \(\hat{A}\) and \(\hat{B}\) associated with the eigenvalues \(a\) and \(b\), respectively. From \([\hat{A}, \hat{C}] = 0\) it follows that \(\hat{C}|a, b\rangle\) is also eigenvectors of \(\hat{A}\) associated with the same eigenvalue \(a\). Therefore, we have

\[ \hat{A} \left( \hat{B}|a, b\rangle \right) = a \left( \hat{B}|a, b\rangle \right), \quad (14.35a) \]

\[ \hat{A} \left( \hat{C}|a, b\rangle \right) = a \left( \hat{C}|a, b\rangle \right), \quad (14.35b) \]

Suppose that \(a\) is a non-degenerate eigenvalue. Then \(\hat{C}|a, b\rangle\) must be proportional to \(\hat{B}|a, b\rangle\), otherwise according to (14.35) \(\hat{C}|a, b\rangle\) and \(\hat{B}|a, b\rangle\)
would be two different eigenvectors associated with the same eigenvalue \( a \), which is incompatible with the hypothesis that \( a \) is a non-degenerate eigenvalue. Therefore, we can write

\[
\hat{C}|a, b\rangle = \gamma \left( \hat{B}|a, b\rangle \right) = \gamma b|a, b\rangle, \tag{14.36}
\]

because \( \hat{B}|a, b\rangle = b|a, b\rangle \), and \( \gamma \) is a numerical constant. Multiplying by \( \hat{B} \) both sides of this equation, we obtain:

\[
\hat{B}\hat{C}|a, b\rangle = \gamma b \left( \hat{B}|a, b\rangle \right) = \gamma b^2|a, b\rangle. \tag{14.37}
\]

On the other hand, applying \( \hat{C} \) to both sides of

\[
\hat{B}|a, b\rangle = b|a, b\rangle, \tag{14.38}
\]

we find

\[
\hat{C}\hat{B}|a, b\rangle = b \left( \hat{C}|a, b\rangle \right) = \gamma b^2|a, b\rangle, \tag{14.39}
\]

where (14.36) has been used. Subtracting (14.39) from (14.37) we obtain

\[
[\hat{B}, \hat{C}]|a, b\rangle = 0. \tag{14.40}
\]

This equality cannot be true for all the basis vectors \( \{|a, b\rangle\} \), because this would imply

\[
[\hat{B}, \hat{C}] = 0, \tag{14.41}
\]

in contradiction to the assumption (14.34). Therefore, the presupposition that the eigenvalue \( a \) is non-degenerate cannot be valid for all eigenvalues of the operator \( \hat{A} \). This means that \( \hat{A} \) has at least one degenerate eigenvalue.

### 14.3 Definitions and notation for the eigenvalues of \( \hat{J}^2 \) and \( \hat{J}_z \)

From the definition (14.2) it follows that for any ket \( |\psi\rangle \), the expectation value \( \langle \psi|\hat{J}^2|\psi\rangle \) is nonnegative, because

\[
\langle \psi|\hat{J}^2|\psi\rangle = \langle \psi|\hat{J}_x^2|\psi\rangle + \langle \psi|\hat{J}_y^2|\psi\rangle + \langle \psi|\hat{J}_z^2|\psi\rangle
= \|\hat{J}_x|\psi\rangle\|^2 + \|\hat{J}_y|\psi\rangle\|^2 + \|\hat{J}_z|\psi\rangle\|^2 \geq 0, \tag{14.42}
\]
14.3. THE EIGENVALUES OF $\hat{J}^2$ AND $\hat{J}_z$

where $\langle \psi | \hat{J}_z^2 | \psi \rangle = \langle \psi | \hat{J}_z \hat{J}_z | \psi \rangle = \| \hat{J}_z | \psi \rangle \|^2$, et cetera (remember that $\hat{J}_z$ is Hermitian, therefore $\langle \psi | \hat{J}_z^\dagger = \langle \psi | \hat{J}_z \rangle$. If $| \psi \rangle$ is an eigenvector of $\hat{J}^2$ associated with the eigenvalue $\lambda \hbar^2$ ($\lambda$ is thus dimensionless), then (14.42) implies

$$\lambda \geq 0. \quad (14.43)$$

It is conventional (but not mandatory) to introduce a nonnegative number $j \geq 0$ defined by

$$\lambda = j (j + 1). \quad (14.44)$$

For $j \geq 0$ the function $j (j + 1)$ is positive or null and monotonically increasing, as shown in Fig. 14.1.

$$\lambda = j(j + 1)$$

![Graph of the function $\lambda = j(j + 1)$ for $j \geq 0$.](image)

Figure 14.1: Graph of the function $\lambda = j(j + 1)$, for $j \geq 0$. The vertical gray line mark the value $j = (\sqrt{5} - 1)/2$, which gives $\lambda = 1$.

Therefore, if necessary, we can invert (14.44) to obtain

$$j = \frac{1}{2} \left( \sqrt{4\lambda + 1} - 1 \right). \quad (14.45)$$

According to Sec. 14.2, since $\hat{J}^2$ and $\hat{J}_z$ commute, it is possible to find a set of common eigenvectors $\{ | j, m \rangle \}$, such that

$$\hat{J}^2 | j, m \rangle = j(j + 1) \hbar^2 | j, m \rangle, \quad (14.46a)$$

$$\hat{J}_z | j, m \rangle = m \hbar | j, m \rangle, \quad (14.46b)$$
and

\[ \langle j', m' | j, m \rangle = \delta_{jj'} \delta_{mm'}, \quad (14.47) \]

where the eigenvalue of \( \hat{J}_z \) is traditionally written as \( m \hbar \), and \( m \) is a dimensionless number\(^4\).

From inspection of (14.1-14.3), we can easily convince ourselves that using only \( \hat{J}_x, \hat{J}_y \) and \( \hat{J}_z \), it is not possible to build an additional (nontrivial) Hermitian operator commuting with both \( \hat{J}^2 \) and \( \hat{J}_z \). Therefore, from the results of Sec. 14.2, it follows that \{\( J^2, J_z \)\} is a CSCO. This can be also rigorously demonstrated using group theory, but this is outside the scope of these lectures. In short, in the language of group theory, the set of the three Hermitian operators \( \hat{J}_x, \hat{J}_y \) and \( \hat{J}_z \), is closed under commutation, as shown by (14.1). These operators are the generators of a Lie group. Specifically, the commutation relations (14.1) represent the Lie algebra of the SO(3) group. The rank of a Lie group is defined as the largest number of generators commuting with each other. Since none of the operators \( \hat{J}_x, \hat{J}_y \) and \( \hat{J}_z \) commute with any other, the SO(3) group has rank 1 and it has a single independent Casimir element\(^5\), which is \( \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \). Therefore, there are three independent pairs of commuting operators: \{\( \hat{J}_x, \hat{J}_z \)\}, \{\( \hat{J}_y, \hat{J}_z \)\} and \{\( \hat{J}_x, \hat{J}_y \)\}. We can choose any one of these pairs to build a CSCO. Traditionally, the choice is \{\( \hat{J}_x, \hat{J}_z \)\}. So, assigning the two eigenvalues of \( \hat{J}^2 \) and \( \hat{J}_z \) (that is, fixing \( j \) and \( m \)), completely defines the eigenvector \( |j, m\rangle \). However, according to the last result of Sec. 14.2, since the \( J^2 \) is compatible with \( J_x, J_y \) and \( J_z \), but \( J_x, J_y \) and \( J_z \) are reciprocally incompatible, we expect the eigenvalue \( j(j + 1)\hbar^2 \) of \( \hat{J}^2 \), to be degenerate.

\(^4\)It should be noted that at this point we are not making any assumption upon the either discrete or continuous nature of the eigenvalues \( j \) and \( m \). Therefore, \( \delta_{jj'} \) and \( \delta_{mm'} \) can be interpreted either as discrete Kronecker’s delta, or continuous Dirac’s delta functions.

\(^5\)In our non-technical parlance, we can say that a Casimir operator is an operator that commutes with \( \hat{J}_x, \hat{J}_y \) and \( \hat{J}_z \).
14.4 Properties of the eigenvalues and eigenvectors of $\hat{J}^2$ and $\hat{J}_z$

Let $j(j+1)\hbar^2$ and $m\hbar$ be the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ associated with the eigenvector $|j,m\rangle$. Then $j$ and $m$ satisfy the following inequality:

$$-j \leq m \leq j. \quad (14.48)$$

To prove this assertion, consider first the two vectors $\hat{J}_+|j,m\rangle$ and $\hat{J}_-|j,m\rangle$. By definition, their squared norms are nonnegative, that is

$$\|\hat{J}_+|j,m\rangle\|^2 = \langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle \geq 0, \quad (14.49a)$$

$$\|\hat{J}_-|j,m\rangle\|^2 = \langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle \geq 0, \quad (14.49b)$$

where we have used $\hat{J}_- = (\hat{J}_+)^\dagger$. Substituting (14.12) and (14.13) in the equations above, we find

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle = \langle j,m|\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z|j,m\rangle$$

$$= (j(j+1) - m(m+1))\hbar^2 \langle j,m|j,m\rangle \geq 0, \quad (14.50a)$$

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = \langle j,m|\hat{J}^2 - \hat{J}_z^2 + \hbar \hat{J}_z|j,m\rangle$$

$$= (j(j+1) - m(m-1))\hbar^2 \langle j,m|j,m\rangle \geq 0. \quad (14.50b)$$

These equations imply

$$j(j+1) - m(m+1) = (j-m)(j+m+1) \geq 0, \quad (14.51a)$$

$$j(j+1) - m(m-1) = (j+m)(j-m+1) \geq 0, \quad (14.51b)$$

where to obtain the right sides we have added the null term $jm - mj = 0$ to the left sides. For example,

$$j(j+1) - m(m+1) = j(j+1) - m(m+1) + jm - mj$$

$$= j(j + m(m+1)) - m(m+1 + j)$$

$$= (j-m)(j+m+1). \quad (14.52)$$
Then, from (14.51a) and (14.51b) it follows that

\[ -(j + 1) \leq m \leq j, \quad (14.53a) \]

\[ -j \leq m \leq j + 1, \quad (14.53b) \]

respectively. From \( j \geq 0 \), it follows that \((j + 1) < -j \leq j < j + 1\). Therefore, equations (14.53a) and (14.53b) are satisfied together only if \( m \)

satisfies the inequality

\[ -j \leq m \leq j. \quad (14.54) \]

This proves (14.48).

A necessary and sufficient condition for a ket \(|\psi\rangle\) to be a null vector, that is \(|\psi\rangle = 0\), is that its norm vanishes: \( \langle\psi|\psi\rangle = 0 \). Therefore, from (14.49a,14.50a,14.51a) it follows that

\[ \hat{J}_+ |j, m\rangle = 0 \quad \text{if and only if} \quad (j - m)(j + m + 1) = 0. \quad (14.55) \]

The solutions of the algebraic equation \((j - m)(j + m + 1) = 0\) are \( m = j \) and \( m = -j - 1 < -j \). Since the values of \( m \) are constrained by \(-j \leq m \leq j\), only the solution \( m = j \) is acceptable and (14.55) implies:

\[ \hat{J}_+ |j, j\rangle = 0. \quad (14.56) \]

Similarly, from (14.49b,14.50b,14.51b) it follows that

\[ \hat{J}_- |j, m\rangle = 0 \quad \text{if and only if} \quad (j + m)(j - m + 1) = 0, \quad (14.57) \]

and, therefore,

\[ \hat{J}_- |j, -j\rangle = 0. \quad (14.58) \]

The converse relations are also true, namely

\[ \hat{J}_\pm |j, m\rangle = 0 \quad \implies \quad m = \pm j. \quad (14.59) \]
14.4. EIGENVALUES AND EIGENVECTORS OF $\hat{J}^2$ AND $\hat{J}_Z$

This is easy to prove. Applying $\hat{J}_\pm$ to both sides of $\hat{J}_\pm |j, m\rangle = 0$ and using (14.13) and (14.51), we obtain

$$\hat{J}_\pm \hat{J}_\pm |j, m\rangle = (\hat{J}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z) |j, m\rangle$$

$$= \hbar^2 [j(j + 1) - m(m \pm 1)] |j, m\rangle$$

$$= \hbar^2 [(j \mp m)(j \pm m + 1)] |j, m\rangle = 0. \quad (14.60)$$

The last equality together with (14.48) implies $m = \pm j$. What if $m \neq \pm j$? Then, the following two statements are true:

1) If $m > -j$, $\hat{J}_- |j, m\rangle$ is a non-null eigenvector of $\hat{J}^2$ and $\hat{J}_z$ associated with the eigenvalues $j(j + 1) \hbar^2$ and $(m - 1) \hbar$, respectively.

2) If $m < j$, $\hat{J}_+ |j, m\rangle$ is a non-null eigenvector of $\hat{J}^2$ and $\hat{J}_z$ associated with the eigenvalues $j(j + 1) \hbar^2$ and $(m + 1) \hbar$, respectively.

To prove 1), first we notice that from (14.49b,14.50b) it follows that $\hat{J}_- |j, m\rangle$ is a non-null vector because its norm is positive for $m > -j$.

Then, we show that $\hat{J}_- |j, m\rangle$ is an eigenvector of $\hat{J}^2$. Using (14.10d), we can write

$$[\hat{J}^2, \hat{J}_-] |j, m\rangle = 0. \quad (14.61)$$

This can be rewritten as

$$\hat{J}^2 (\hat{J}_- |j, m\rangle) = \hat{J}_- \hat{J}^2 |j, m\rangle$$

$$= j(j + 1) \hbar^2 \left( \hat{J}_- |j, m\rangle \right), \quad (14.62)$$

which signifies that $\hat{J}_- |j, m\rangle$ is an eigenvector of $\hat{J}^2$ associated with the eigenvalue $j(j + 1) \hbar^2$.

Now, we prove that $\hat{J}_- |j, m\rangle$ is an eigenvector of $\hat{J}_z$. If we multiply from right both sides of (14.10b) by $|j, m\rangle$, we obtain

$$[\hat{J}_z, \hat{J}_-] |j, m\rangle = -\hbar \hat{J}_- |j, m\rangle, \quad (14.63)$$
which is equivalent to

\[
\hat{J}_z \left( \hat{J}_- |j, m\rangle \right) = \hat{J}_- \hat{J}_z |j, m\rangle - \hbar \hat{J}_- |j, m\rangle
\]

\[
= (m - 1)\hbar \left( \hat{J}_- |j, m\rangle \right) . \tag{14.64}
\]

This shows that \( \hat{J}_- |j, m\rangle \) is an eigenvector of \( \hat{J}_z \) with the eigenvalue \((m - 1)\hbar\).

If \( m < j \), we can use arguments similar to the ones leading to (14.62) and (14.64), to prove statement 2). Of course, we must replace \( \hat{J}_- |j, m\rangle \) with \( \hat{J}_+ |j, m\rangle \), work again with (14.10d) and use (14.10a) instead of (14.10b). The final result is:

\[
\hat{J}_z \left( \hat{J}_+ |j, m\rangle \right) = (m + 1)\hbar \left( \hat{J}_+ |j, m\rangle \right) . \tag{14.65}
\]

These relations may be straightforwardly generalized to include powers of \( \hat{J}_\pm \), that is \((\hat{J}_\pm)^n\), where \( n = 1, 2, \ldots \) is a positive integer. Specifically, we assert that

3) If \(-j + n \leq m \leq j\), then \((\hat{J}_-)^n |j, m\rangle\) is an eigenvector of \( \hat{J}^2 \) and \( \hat{J}_z \) associated with the eigenvalues \( j(j + 1)\hbar^2 \) and \((m - n)\hbar\), respectively.

4) If \(-j \leq m \leq j - n\), then \((\hat{J}_+)^n |j, m\rangle\) is an eigenvector of \( \hat{J}^2 \) and \( \hat{J}_z \) associated with the eigenvalues \( j(j + 1)\hbar^2 \) and \((m + n)\hbar\), respectively.

The proof is very simple. First, we use (14.23) to write:

\[
[\hat{J}^2, (\hat{J}_\pm)^n] |j, m\rangle = 0, \tag{14.66}
\]

which can be rewritten as

\[
\hat{J}^2 \left[ (\hat{J}_\pm)^n |j, m\rangle \right] = (\hat{J}_\pm)^n \hat{J}^2 |j, m\rangle
\]

\[
= j(j + 1)\hbar^2 \left[ (\hat{J}_\pm)^n |j, m\rangle \right] . \tag{14.67}
\]
14.5. **THE SPECTRUM OF $\hat{J}^2$ AND $\hat{J}_z$**

This relation means that $(\hat{J}_\pm)^n|j, m\rangle$ is an eigenvector of $\hat{J}^2$ with eigenvalue $j(j + 1)\hbar^2$. Next, if we multiply from right both sides of equations (14.15) by $|j, m\rangle$, we obtain

$$[\hat{J}_z, (\hat{J}_\pm)^n]|j, m\rangle = \pm n\hbar (\hat{J}_\pm)^n|j, m\rangle,$$

(14.68)

namely,

$$\hat{J}_z[(\hat{J}_\pm)^n|j, m\rangle] = (\hat{J}_\pm)^n\hat{J}_z|j, m\rangle \pm n\hbar(\hat{J}_\pm)^n|j, m\rangle$$

$$= m\hbar (\hat{J}_\pm)^n|j, m\rangle \pm n\hbar(\hat{J}_\pm)^n|j, m\rangle$$

$$= (m \pm n)\hbar [(\hat{J}_\pm)^n|j, m\rangle].$$

(14.69)

Therefore, $(\hat{J}_\pm)^n|j, m\rangle$ is an eigenvector of $\hat{J}_z$ with eigenvalue $(m \pm n)\hbar$. However, since (14.48) requires that $-j \leq m \leq j$, we must demand

$$-j \leq m + n \leq j \iff -j \leq m \leq j - n,$$

(14.70a)

$$-j \leq m - n \leq j \iff -j + n \leq m \leq j.$$  

(14.70b)

This concludes the demonstration of statements 3) and 4).

### 14.5 The spectrum of $\hat{J}^2$ and $\hat{J}_z$

Now we are able to determine the possible values of $j$ and $m$, that is, the spectrum of $\hat{J}^2$ and $\hat{J}_z$. According to our previous findings, since the set $\{\hat{J}^2, \hat{J}_z\}$ is a CSCO, the knowledge of $j$ and $m$ uniquely identify the eigenvector $|j, m\rangle$ common to $\hat{J}^2$ and $\hat{J}_z$. Therefore, once we know the spectrum of $\hat{J}^2$ and $\hat{J}_z$, we also know the common eigenvectors. Thus, let $j(j + 1)\hbar^2$ and $m\hbar$ be the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ associated with the eigenvector $|j, m\rangle$. We do not make any hypothesis about the values of $j$ and $m$; we only require, according to (14.48), that $j$ and $m$ satisfy the inequality $-j \leq m \leq j$, with $j \geq 0$. So, at this stage $m$ can be any real number between $-j$ and $j$. 
Consider the two vectors \((\hat{J}_+)^p|j, m\rangle\) and \((\hat{J}_-)^q|j, m\rangle\), where \(p\) and \(q\) are nonnegative integers. According to 3) of the previous section, \((\hat{J}_-)^q|j, m\rangle\) is an eigenvector of \(\hat{J}^2\) and \(\hat{J}_z\) with the eigenvalues \(j(j+1)\hbar^2\) and \((m-q)\hbar\), respectively. Similarly, from 4) it follows that \((\hat{J}_+)^p|j, m\rangle\) is an eigenvector of \(\hat{J}^2\) and \(\hat{J}_z\) with the eigenvalues \(j(j+1)\hbar^2\) and \((m+p)\hbar\), respectively. As usual, (14.48) requires that

\[
-j \leq m - q, \quad (14.71a)
\]

\[
m + p \leq j. \quad (14.71b)
\]

Now, let us choose \(p\) and \(q\) to be the greatest nonnegative integers such that:

\[
m - (q + 1) < -j, \quad (14.72a)
\]

\[
m + (p + 1) > j, \quad (14.72b)
\]

as illustrated in Fig. 14.2.

![Graphical representation of the inequalities (14.72). By hypothesis, the nonnegative integers \(q\) and \(p\) are chosen to satisfy \(a = j+(m-q) < 1\) and \(b = j-(m+p) < 1\). The intervals \(a\) and \(b\) are pictured as gray bands.](image)

The unnormalized eigenvectors of \(\hat{J}^2\) and \(\hat{J}_z\) with the eigenvalues of \(\hat{J}_z\) proportional to

\[
m - q, \ldots, m - 1, m, m + 1, \ldots, m + p, \quad (14.73)
\]

are, according to 3) and 4) of the previous section,

\[
(\hat{J}_-)^q|j, m\rangle, \ldots, \hat{J}_-|j, m\rangle, |j, m\rangle, \hat{J}_+|j, m\rangle, \ldots, (\hat{J}_+)^p|j, m\rangle. \quad (14.74)
\]
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It is important to understand that these are the only possible vectors with these properties, because the pair $(j, m)$ uniquely identify (up to an irrelevant multiplicative numerical constant) the eigenvector $|j, m\rangle$. Given the leftmost and the rightmost vectors $(\hat{J}_-)^{q+1}|j, m\rangle$ and $(\hat{J}_+)^{p+1}|j, m\rangle$, we can calculate

$$\hat{J}_z[(\hat{J}_-)^{q+1}|j, m\rangle] = (m - q - 1)\hbar [(\hat{J}_-)^{q+1}|j, m\rangle],$$  \hspace{1cm} (14.75a)

$$\hat{J}_z[(\hat{J}_+)^{p+1}|j, m\rangle] = (m + p + 1)\hbar [(\hat{J}_+)^{p+1}|j, m\rangle].$$  \hspace{1cm} (14.75b)

According to (14.75a), either $$(\hat{J}_-)^{q+1}|j, m\rangle = 0$$ or $$(\hat{J}_-)^{q+1}|j, m\rangle$$ is an eigenvector of $\hat{J}_z$ with eigenvalue $m - q - 1$. However, from (14.72) it follows that $m - q - 1 < -j$, in contradiction with (14.48). Therefore, we conclude that $$(\hat{J}_-)^{q+1}|j, m\rangle = 0.$$ An analogous argument yields to $$(\hat{J}_+)^{p+1}|j, m\rangle = 0.$$

When a vector is null, its norm is equal to zero, that is

$$\| (\hat{J}_-)^{q+1}|j, m\rangle \|^2 = \langle j,m|(\hat{J}_+)^{q+1}(\hat{J}_-)^{q+1}|j,m\rangle = 0,$$  \hspace{1cm} (14.76a)

$$\| (\hat{J}_+)^{p+1}|j, m\rangle \|^2 = \langle j,m|(\hat{J}_-)^{p+1}(\hat{J}_+)^{p+1}|j,m\rangle = 0.$$  \hspace{1cm} (14.76b)

It is convenient to rewrite these equations as:

$$\langle j,m|(\hat{J}_+)^{q+1}(\hat{J}_-)^{q+1}|j,m\rangle = \langle j,m|(\hat{J}_+)^{q}(\hat{J}_+\hat{J}_-)(\hat{J}_-)^{q}|j,m\rangle = 0,$$  \hspace{1cm} (14.77a)

$$\langle j,m|(\hat{J}_-)^{p+1}(\hat{J}_-)^{p+1}|j,m\rangle = \langle j,m|(\hat{J}_-)^{p}(\hat{J}_-\hat{J}_+)(\hat{J}_+)^{p}|j,m\rangle = 0.$$  \hspace{1cm} (14.77b)

Substituting (14.12) and (14.13) in the equations above, we find

$$\langle \alpha |\hat{J}_+\hat{J}_-|\alpha \rangle = \langle \alpha |\hat{J}^2 - \hat{J}_z^2 + \hbar \hat{J}_z|\alpha \rangle$$

$$= \left[j(j + 1) - (m - q)(m - q - 1)\right]\hbar^2 \langle \alpha |\alpha \rangle$$

$$= 0,$$  \hspace{1cm} (14.78a)

$$\langle \beta |\hat{J}_-\hat{J}_+|\beta \rangle = \langle \beta |\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z|\beta \rangle$$

$$= \left[j(j + 1) - (m + p)(m + p + 1)\right]\hbar^2 \langle \beta |\beta \rangle$$

$$= 0.$$  \hspace{1cm} (14.78b)
where we have used the shorthand notation,

$$|\alpha\rangle \equiv (\hat{J}_-)^q|j, m\rangle, \quad |\beta\rangle \equiv (\hat{J}_+)^p|j, m\rangle. \quad (14.79)$$

Equations (14.78) imply that:

$$j(j + 1) - (m - q)(m - q - 1) = (j + m - q)(j - m + q + 1) = 0, \quad (14.80a)$$

$$j(j + 1) - (m + p)(m + p + 1) = (j - m - p)(j + m + p + 1) = 0. \quad (14.80b)$$

The solution of these two algebraic equations are:

$$m = q - j, \quad \text{or} \quad m = j + (q + 1) > j, \quad (14.81a)$$

$$m = j - p, \quad \text{or} \quad m = -j - (p + 1) < -j. \quad (14.81b)$$

According to the condition (14.48), the only acceptable solutions are:

$$m = q - j \quad \iff \quad m - q = -j, \quad (14.82a)$$

$$m = j - p \quad \iff \quad m + p = j. \quad (14.82b)$$

Clearly, these two equalities are satisfied simultaneously only if

$$q - j = j - p \quad \iff \quad j = \frac{q + p}{2}. \quad (14.83)$$

Therefore, $j$ is equal to a positive or zero integer (by hypothesis $q$ and $p$ are nonnegative integers) divided by 2:

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \quad (14.84)$$

Equations (14.82) imply that if $j$ is an integer, all $m$ values are integers; if $j$ is a half-integer, all $m$ values are half-integers. The allowed values of $m$ for a given $j$ are therefore:

$$-j = m - q, \quad m - q + 1, \ldots, m + p - 1, \quad m + p = j, \quad (14.85)$$

that is

$$m = -j, -j + 1, \ldots, j - 1, j. \quad (14.86)$$

2$j + 1$ values
14.6. MATRIX ELEMENTS OF ANGULAR-MOMENTUM OPERATORS

This equation expresses the fact that each eigenvalue \( j(j + 1)\hbar^2 \) of \( \hat{J}^2 \), is 
\( 2j + 1 \) times degenerate. We predicted this degeneracy at the end of Sec. 14.3, on the ground of commutation relations (14.1).

To illustrate the procedure of building the set \( \{|j,m\}\) of \( 2j + 1 \) eigenvectors of \( \hat{J}^2 \) and \( \hat{J}_z \) with a given \( j \), suppose to fix a value of \( j \) and to consider the eigenvector \( |j, -j\rangle \) with \( m = -j \). This implies \( q = 0 \) and \( p = 2j \), that is one needs \( 2j \) steps to reach the other side of the spectrum with \( m = j \). Therefore, the eigenvalues of \( \hat{J}_z \) are:

\[
m = -j, -j + 1, \ldots, j - 1, j,
\]
and the associated (unnormalized) eigenvectors of \( \hat{J}^2 \) and \( \hat{J}_z \), are obtained by applying \( 2j \) times\(^6\) the ladder operator \( \hat{J}_+ \) to \( |j, -j\rangle \):

\[
|j, -j\rangle, \ \hat{J}_+ |j, -j\rangle, \ \ldots, (\hat{J}_+)^{2j-1} |j, -j\rangle, \ (\hat{J}_+)^{2j} |j, -j\rangle.
\]

Vice versa, if we start from the eigenvector \( |j, j\rangle \) with \( m = j \), then we have \( q = 2j \) and \( p = 0 \). In this case, obviously the eigenvalues of \( \hat{J}_z \) are still

\[
m = -j, -j + 1, \ldots, j - 1, j,
\]
and the associated (unnormalized) eigenvectors of \( \hat{J}^2 \) and \( \hat{J}_z \), are obtained by applying \( 2j \) times the ladder operator \( \hat{J}_- \) to \( |j, j\rangle \):

\[
(\hat{J}_-)^{2j} |j, j\rangle, \ (\hat{J}_-)^{2j-1} |j, j\rangle, \ \ldots, \ \hat{J}_- |j, j\rangle, \ |j, j\rangle.
\]

14.6 Matrix elements of angular-momentum operators

Let us assume that the kets \( |j, m\rangle \) are normalized:

\[
\langle j', m'|j, m \rangle = \delta_{jj'} \delta_{mm'}.
\]

\(^6\)Note that from (14.83) it follows that the number \( 2j = q+p \) is necessarily a nonnegative integer.
Then, from equations (14.46) we clearly have
\[ \langle j', m' | \hat{J}^2 | j, m \rangle = j(j + 1)\hbar^2 \delta_{jj'} \delta_{mm'}, \quad (14.92) \]
and
\[ \langle j', m' | \hat{J}_z | j, m \rangle = m\hbar \delta_{jj'} \delta_{mm'}. \quad (14.93) \]

To find the matrix elements of \( \hat{J}_x \) and \( \hat{J}_y \), we first calculate the matrix elements of \( \hat{J}_\pm \). We have from (14.60)
\[ \langle j, m | \hat{J}_\pm | j, m \rangle = \hbar^2 \left[ (j \mp m)(j \pm m + 1) \right]. \quad (14.94) \]
Since \( \hat{J}_\pm | j, m \rangle \) is an eigenvector of \( \hat{J}_z \) with eigenvalue \((m \mp 1)\hbar\), then it must be the same as \( |j, m \pm 1\rangle \) up to a multiplicative constant\(^7\). Thus, we set
\[ \hat{J}_\pm | j, m \rangle = c^\pm_{jm} | j, m \pm 1\rangle, \quad (14.95) \]
where the coefficients \( c^\pm_{jm} \) have to be determined. Substituting this expression in the left side of (14.94), we obtain
\[ |c^\pm_{jm}|^2 = \hbar^2 \left[ (j \mp m)(j \pm m + 1) \right]. \quad (14.96) \]
This equation determines \( c^\pm_{jm} \) up to an arbitrary phase factor. Conventionally, this phase is chosen to be \textbf{zero}. This choice guarantees that the matrix elements of \( \hat{J}_\pm \) are all nonnegative; this is called the 
\textit{Condon-Shortley phase convention}. So, we found
\[ \hat{J}_\pm | j, m \rangle = \sqrt{(j \mp m)(j \pm m + 1)} \hbar | j, m \pm 1\rangle. \quad (14.97) \]
Therefore, the matrix elements of \( \hat{J}_\pm \) are
\[ \langle j', m' | \hat{J}_\pm | j, m \rangle = \sqrt{(j \mp m)(j \pm m + 1)} \hbar \delta_{jj'} \delta_{m\pm1,m'}. \quad (14.98) \]
\(^7\)Remember that by definition, the eigenvalues \( a, b, \ldots, z \), of the operators \( \hat{A}, \hat{B}, \ldots, \hat{Z} \) corresponding to a CSCO \( \{A, B, \ldots, Z\} \), uniquely determine the eigenvectors \( |a, b, \ldots, z\rangle \) up to a multiplicative constant. Therefore, since \( \hat{J}_\pm | j, m \rangle \) and \( |j, m \pm 1\rangle \) are associated with the same pair of eigenvalues \( j(j + 1)\hbar^2 \) and \( (m \pm 1)\hbar \), it must be \( \hat{J}_\pm | j, m \rangle \propto |j, m \pm 1\rangle. \)
14.6. MATRIX ELEMENTS OF ANGULAR-MOMENTUM OPERATORS

Finally, we determine the matrix elements of \( \hat{J}_x \) and \( \hat{J}_y \) to be

\[
\langle j', m' | \hat{J}_x | j, m \rangle = \frac{1}{2} \left[ \langle j', m' | \hat{J}_+ | j, m \rangle + \langle j', m' | \hat{J}_- | j, m \rangle \right],
\]

(14.99)

and

\[
\langle j', m' | \hat{J}_y | j, m \rangle = \frac{1}{2i} \left[ \langle j', m' | \hat{J}_+ | j, m \rangle - \langle j', m' | \hat{J}_- | j, m \rangle \right].
\]

(14.100)