Addendum to Exercises 11.05.2017

Framework of the problem

This problem is based upon a mathematical model developed by H. Beker\textsuperscript{1}. Consider the linear space $F[x]$ over the field $\mathbb{C}$ of polynomials of degree $\leq n$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$ \hfill (1)

The derivative of $f(x)$,

$$(\hat{D}f)(x) \equiv \frac{df}{dx}(x) = a_1 + 2a_2 x + \ldots + n a_n x^{n-1},$$ \hfill (2)

still belongs to $F[x]$. However, multiplication by $x$ yields a polynomial

$$xf(x) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_n x^{n+1},$$ \hfill (3)

which does not belong to this linear space anymore. To overcome this problem, we define a pseudo position operator as:

$$(\hat{x}f)(x) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_{n-1} x^n.$$ \hfill (4)

This operator multiplies by $x$ the polynomial and truncates it to the term proportional to $x^n$. Of course, this definition is completely arbitrary and other choices could be made.

Following Beker, we agree to represent $f(x)$ in $\mathbb{C}^{n+1}$ by a $(n + 1)$-dimensional vector whose components are the coefficients of the polynomial in the basis $\{1, x, x^2, \ldots, x^n\}$,

$$f(x) := |f\rangle \equiv (a_0, a_1, \ldots, a_n),$$ \hfill (5)

which reads as “$f(x)$ is represented by $|f\rangle \equiv (a_0, a_1, \ldots, a_n)$”. For example, the second degree polynomial $f(x) = a_0 + a_1 x + a_2 x^2$ is represented by

$$|f\rangle = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$ \hfill (6)

Then, operators like $\hat{D}$ or $\hat{x}$ can be represented in $\mathbb{C}^{n+1}$ by $(n + 1) \times (n + 1)$ matrices. These matrices can be easily determined as follows. Consider first the derivative operator $\hat{D}$: its action upon the polynomial $f(x)$ is defined by

$$(\hat{D}f)(x) = g(x),$$ \hfill (7)

\textsuperscript{1}H. Beker, “Special polynomials by matrix algebra,” American Journal of Physics Vol. 66, pp. 812-
where
\[ g(x) = \frac{df}{dx}(x) = a_1 + 2a_2x + \ldots + n a_n x^{n-1}. \quad (8) \]

Now, if we agree to write
\[ g(x) = \sum_{k=0}^{n} b_k x^k, \quad (9) \]
then (8) can be rewritten as:
\[ b_0 + b_1 x + \ldots + b_{n-1} x^{n-1} + b_n x^n = a_1 + 2a_2 x + \ldots + n a_n x^{n-1}. \quad (10) \]

By equating the coefficients of equal powers in \( x \), we find the \( n + 1 \) relations:
\[ \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ \vdots \\ na_n \\ 0 \end{bmatrix}. \quad (11) \]

If with \( D = [D_{kl}] \) we denote the matrix representing \( \hat{D} \), then (11) can be written as,
\[ b_k = \sum_{l=0}^{n} D_{kl} a_l = \begin{cases} (k + 1) a_l, & k = l - 1, \\ 0, & k = n, \end{cases} \quad (12) \]

This equation implies that,
\[ D_{kl} = l \delta_{l,k+1}, \quad (k, l = 0, \ldots, n). \quad (13) \]

Note that for \( k = n \) this equation correctly gives \( D_{nl} = 0 \) because \( \delta_{l,n+1} = 0 \) for all \( l = 0, \ldots, n \). As an example, if \( n = 3 \),
\[ D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (14) \]
and
\[ D[f] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix}. \quad (15) \]
This is consistent with
\[ \frac{d}{dx} \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 \right) = a_1 + 2a_2 x + 3a_3 x^2. \] (16)

To determine \((n + 1) \times (n + 1)\) matrix \(X = [X_{kl}]\) representing the position operator \(\hat{x}\), we can proceed in exactly in the same manner. From \((\hat{x}f)(x) = g(x)\), we find
\[ b_0 + b_1 x + \ldots + b_{n-1} x^{n-1} + b_n x^n = a_0 x + a_1 x^2 + \ldots + a_{n-1} x^n. \] (17)

From this equation it follows that,
\[ b_k = \sum_{l=0}^{n} X_{kl} a_l = \begin{cases} 0, & k = 0; \\ a_l, & k - 1 = l. \end{cases} \Rightarrow X_{kl} = \delta_{l,k-1}, \quad (k,l = 0,\ldots,n). \] (18)

Note that the rightmost equation correctly implies that if \(k = 0\), then \(X_{0l} = 0\) because \(\delta_{l,-1} = 0\) for all \(l = 0,\ldots,n\). For example, for \(n = 3\), we have
\[ X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \] (19)
and
\[ X|f\rangle = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \end{bmatrix}. \] (20)

From the derivative operator \(\hat{D}\), we build the momentum operator \(\hat{p} \equiv -i\hbar \hat{D}\) and the corresponding matrix \(P\). A straightforward calculation for any finite \(n\), shows that the matrices \(P\) and \(X\) are not Hermitian. Moreover, it is also not difficult to verify that for the commutator matrix \(C = XP - PX = [C_{kl}]\), we have
\[ C_{kl} = i\hbar \left[ \delta_{kl} - (n + 1)\delta_{ko}\delta_{ln} \right], \quad (k,l = 0,\ldots,n). \] (21)

For example, for \(n = 3\),
\[ [X, P] = i\hbar \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}. \] (22)

This equation shows that the canonical commutation relations (CCR) \([\hat{x}, \hat{p}] = i\hbar\) cannot be satisfied by finite-dimensional matrices. It is also clear that the last diagonal term
\(C_{33} = -3i\hbar\) serves to make \(\text{tr} ([X, P]) = 0\). Similarly, it is not difficult to show that the anti-commutator matrix \(A = XP + PX = [A_{kl}]\), has the following elements:

\[
A_{kl} = -i\hbar \left[ (2k + 1)\delta_{kl} - (n + 1)\delta_{kn}\delta_{ln} \right], \quad (k, l = 0, \ldots, n). \tag{23}
\]

For example, for \(n = 3\),

\[
\{X, P\} = -i\hbar \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}. \tag{24}
\]

**Statement of the problem**

The uncertainty relation for the matrices \(X\) and \(P\) reads as (see “Addendum to Lecture 7”):

\[
\delta X_f \delta P_f \geq \frac{1}{2} \left| \langle f|\Delta X_f^{\dagger}\Delta P_f - \Delta P_f^{\dagger}\Delta X_f|f\rangle \right|, \tag{25}
\]

where we have defined:

\[
\delta X_f \equiv \sqrt{\langle f|\Delta X_f^{\dagger}\Delta X_f|f\rangle}, \quad \text{with} \quad \Delta X_f \equiv X - \langle X \rangle_f I, \quad \text{and} \quad \langle X \rangle_f \equiv \langle f|X|f\rangle, \tag{26}
\]

where \(I\) denotes the \((n + 1) \times (n + 1)\) identity matrix, and

\[
\delta P_f \equiv \sqrt{\langle f|\Delta P_f^{\dagger}\Delta P_f|f\rangle}, \quad \text{with} \quad \Delta P_f \equiv P - \langle P \rangle_f I, \quad \text{and} \quad \langle P \rangle_f \equiv \langle f|P|f\rangle. \tag{27}
\]

Solve *numerically* the set of \(n + 1\) nonlinear algebraic equations,

\[
(X - \langle X \rangle_f)|f\rangle = -\frac{\langle f|\Delta X_f^{\dagger}\Delta P_f - \Delta P_f^{\dagger}\Delta X_f|f\rangle}{2\langle f|\Delta P_f^{\dagger}\Delta P_f|f\rangle} (P - \langle P \rangle_f)|f\rangle, \tag{28}
\]

to determine the \(n + 1\) components \((a_0, a_1, \ldots, a_n)\) of the vector

\[
|f\rangle = \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_n
\end{bmatrix}, \tag{29}
\]

for \(n = 2, 3, 4\) (and more, if you can), and show that for such \(|f\rangle\)'s the inequality (25) becomes an identity.