- Postulate 0 -

"Let \( f \) be a physical quantity.

"If in the system prepared in the state \( \psi_1(q) \) the measurement of \( f \) gives ALWAYS \( f_1 \), and in the system prepared in \( \psi_2(q) \) the measurement of \( f \) ALWAYS gives \( f_2 \), then any linear combination

\[
\psi(q) = a \psi_1(q) + c \psi_2(q)
\]

represents a state for which the measurement of \( f \) gives EITHER \( f_1 \) OR \( f_2 \)."

This is called:

Principal of superposition of the states

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Postulate I -

If \( f \) is a physical quantity, the value that can assume are either discrete \( \psi_1, \psi_2, \ldots, \psi_n \) or continuous \( f(x) \). Consider discrete first.

Let \( \psi_n(q) \) be the wave function of the system when \( f \) has the value \( \psi_n \). Then there exist an operator \( \hat{f} \) associated with \( f \) such that:

\[
\hat{f} \psi_n(q) = f_n \psi_n(q)
\]

If \( \lambda \in \mathbb{R} \) we can always write:

\[
\lambda (\hat{f} \psi_n(q)) = \lambda (f_n \psi_n(q))
\]

\( \Rightarrow \hat{f} \psi_n(q) = \lambda \psi_n(q) \) when \( \hat{f} \psi_n = \lambda \psi_n \)

We can always choose \( \lambda \) such that:

\[
\int |\psi_n|^2 \,dq = 1
\]

Postulate II -

If the system is prepared in the state represented by \( \psi(q) \) and then a measurement of \( f \) yields the value \( f_n \) and leave the system in the state \( \psi_n \).

\[
\psi(q) \xrightarrow{\text{measurement of } f} \psi_n(q)
\]

This implies that there is a non-zero probability to find the system prepared in the state \( \psi_n \) in the state \( \psi_n \).
This means that $Y(x)$ can be written as a linear combination of $Y_n$:

$$Y = \sum_n a_n Y_n$$

and when a measurement of $f$ is made and $f_n$ found, then

$$a_n \to 0 \text{ unless } n = k$$

If such a lin. comb. is possible, the system of

$$\{Y_1, Y_2, \ldots, Y_n, \ldots\}$$

is said COMPLETE.

From $Y = \sum_n a_n Y_n$ we can find the probability to obtain $f_n$ from a measurement of $f$.

**Postulate III**

If a system is in the state $Y(x, t)$, the average value of $f$ at time $t$ is:

$$\langle f \rangle_Y = \int Y^*(\hat{f} Y) \, dq$$

$$\langle f \rangle_Y = \text{expectation value}$$

By def., if $Y = Y_n$ then

$$\langle f \rangle_{Y_n} = \int Y_n^* (\hat{f} Y_n) \, dq = f_n Y_n$$

$$= f_n \int Y_n^* Y_n \, dq = f_n$$

However, from the def. of probability:

$$\langle f \rangle_Y = \sum_n f_n P(f_n)$$
where \( P(f_n) = \text{probability to obtain } f_n \text{ from a measurement of } f \text{ in the system prepared in } \hat{\Psi} \)

By comparison,

\[
\langle f \rangle_\hat{\Psi} = \sum_n f_n P(f_n) = \int \hat{\Psi}^* (\hat{f} \hat{\Psi}) \, dq
\]

we use \( \hat{\Psi} = \sum_n \alpha_n \hat{\Psi}_n \) to write:

\[
\sum_n f_n P(f_n) = \sum_{k,e} \alpha_k^* \alpha_e \int \hat{\Psi}_k^* \hat{\Psi}_e \, dq
\]

\[
= \sum_{k,e} \alpha_k^* \alpha_e \int \hat{\Psi}_k^* \hat{\Psi}_e \, dq
\]

\[
= \sum_{k,e} \beta_k^* \beta_e \left[ \sum_{k} \alpha_k^* \alpha_k \int \hat{\Psi}_k^* \hat{\Psi}_k \, dq \right]
\]

Therefore, we would identify \( f_n \) to be:

\[
P(f_n) = \sum_k \alpha_k^* \alpha_n \int \hat{\Psi}_k^* \hat{\Psi}_n \, dq \geq 0
\]

For arbitrary \( \hat{\Psi} \), this is possible only if orthogonality:

\[
\int \hat{\Psi}_k^* \hat{\Psi}_n \, dq = 0 \quad \text{or} \quad \sum_k \alpha_k^* \alpha_n \int \hat{\Psi}_k^* \hat{\Psi}_n \, dq = 0
\]

\[
\Rightarrow P(f_n) = \sum_k \alpha_k^* \alpha_n \sum_k \alpha_k = 12n^2 \geq 0
\]

So:

\[
\langle f \rangle_\hat{\Psi} = \sum_n f_n 12n^2
\]
\[ \int y^* y = 1 \]

It follows

\[ \sum_{n,m} \delta_{nm} \int y^*_n y_m dq = 1 \]

\[ \sum |a_n|^2 = 1 \quad \text{sum of probability} = 1 \]

**How to find \( a_n \)?**

From \( Y = \sum_n a_n y_n \) \( \Rightarrow \)

\[ y^* = \sum_n a_n^* y_n^* \]

Then

\[ \int |y|^2 dq = \sum_n |a_n|^2 \int y y^*_n dq \]

On the other hand

\[ \int |y|^2 dq = \sum_n a_n^* a_n = 1 \]

\[ \Rightarrow \]

\[ a_n = \int y_n^* y dq \]

If this is a **scalar** product, then

\[ a_n = \langle y_n | y \rangle \]

The completeness relation can be written as:

\[ y(q) = \sum_n a_n y_n(q) \quad \text{but} \quad a_n = \int y_n^*(q') y(q') dq' \]

\[ \Rightarrow y(q) = \sum_n \int y_n^*(q') y(q') dq' y_n(q) = \int dq' \left[ \sum_n y_n(q) y_n^*(q') \right] y(q') dq' \]

However, by def: \( y(q) = \int \delta(q-q') y(q') dq' \quad \forall y \Rightarrow \)

\[ \sum_n y_n(q) y_n^*(q') = \delta(q-q') \quad \text{closed} \]
Note, this can be written as:
\[ \sum_n \langle 14_n | 14 \rangle = 1 \]

Then
\[ |14\rangle = \frac{1}{\sqrt{2}} (|14\rangle + |24\rangle) = \sum_n \langle 14_n | 14 \rangle = \sum_n \langle 14_n | 14 \rangle \]

\[ \langle 14_n | 14 \rangle = \int \psi_{14}^*(q) \psi_{14}(q) dq \]

**Consequences of the Postulates**

An operator \( \hat{F} \) has always a linear integral representation.

Let
\[ \hat{F} \psi = \sum_n \hat{F} \psi_n \]
\[ = \sum_n \langle \psi_n | \psi \rangle \psi_n \]

but \( \psi_n = \sum\int \psi_{14}^* \psi_{14} dq = \sum \langle \psi_{14}^* | \psi_{14} \rangle = \sum \int \psi_{14}^*(q') \psi_{14}(q') dq' \psi_{14}(q) = \int \sum [\psi_{14}^*(q') \psi_{14}(q)] \psi_{14}(q') dq' \]

\[ \Rightarrow \hat{F} \psi(q) = \int K(q, q') \psi(q') dq' \]

where
\[ K(q, q') = \sum \psi_{14}^*(q') \psi_{14}(q) \]

\[ K(q, q') \] is the kernel of the operator.
- Hermitian operators -

For a physical quantity it must be

\[ \langle \phi | \psi \rangle \in \mathbb{R} \quad \forall \phi, \psi \]

So \( \langle \phi | \psi \rangle = \int \phi^* (\hat{\psi} \phi) \, dq = (\langle \psi | \phi \rangle)^* = \int \psi^* (\hat{\phi} \psi) \, dq \)

Def. of conj. op.: if \( (\hat{\phi} \psi) = \phi \) then \( \hat{\phi}^* (\hat{\psi} \phi) = \phi^* \)

Now, the transpose of an operator \( \hat{\phi}^T \) is defined so:

\[ \hat{\phi}^T = \int \phi(q) (\hat{\psi} \phi)(q) \, dq = \int \psi(q) (\hat{\phi}^T \phi)(q) \, dq \]

\( \forall \phi, \psi \)

If in this def. I take \( \phi = \psi^* \) \( \Rightarrow \)

\[ \langle \phi | \psi \rangle = \int \phi^* (\hat{\psi} \phi) \, dq = \int \psi^* (\hat{\phi} \psi) \, dq \]

Comparisons of \( 1 \) and \( 2 \) show that it must be:

\[ \int \psi^* (\hat{\phi} \psi) \, dq = \int \phi^* (\hat{\psi} \phi) \, dq \Rightarrow \]

\[ \hat{\phi}^T = \hat{\psi}^* \]

for an observable

Taking the conjugate of both:

\[ (\hat{\phi}^T)^* = \hat{\phi} \]
Def: \[ \hat{\varphi}^+ = (\hat{\varphi}^T)^* = (\hat{\varphi}^*)^T \]

Then if \[ \hat{\varphi} = \hat{\varphi}^+ \Rightarrow \hat{\varphi} \text{ is Hermitian} \]

- Example -

From \[ (\hat{\varphi}_\psi)(q) = \int K(q, q') \psi(q') dq' \]

it follows

\[ \langle \varphi \rangle_\psi = \int \psi^*(\hat{\varphi}_\psi) dq \]

\[ = \int \psi^*(q) K(q, q') \psi(q') dq' dq' \]

\[ = \int \psi^*(q') K^*(q', q) \psi(q) dq' dq \]

\[ = \int \psi^*(q) K^*(q', q) \psi(q') dq dq' \]

they are equal if

\[ K(q, q') = K^*(q', q) \]

If \[ K(q, q') = \sum_n f_n \psi^*_n(q') \psi_n(q) \Rightarrow K^*(q', q) = \sum_n f^*_n \psi_n(q) \psi^*_n(q') \]

\[ \Rightarrow f_n = f^*_n \]

\[ \Rightarrow \text{The eigenvalues of a Hermitian operator are real!} \]
The eigenfunction of an Hermitian operator corresponding to different eigenvalues are orthogonal.

- dim:

Let \( \hat{P} \phi_n = f_n \phi_n \) and \( \hat{P} \phi_m = f_m \phi_m \) \( \Rightarrow \hat{P}^{\dagger} \phi^* m = f_m \phi^* m \)

Then

\( \phi^* m \hat{P} \phi_n = f_n \phi^* m \phi_n \)

Subtract:

\( \phi^* n \hat{P} \phi^* m = f_m \phi^* n \phi^* m \)


\( \phi^* m \hat{P} \phi_n - \phi^* n \hat{P} \phi^* m = (f_n - f_m) \phi^* m \phi_n \)

Integrate both sides:

\[
\int [\phi^* m (\hat{P} \phi_n) - \phi^* n (\hat{P} \phi^* m)] \, dq = (f_n - f_m) \int \phi^* m \phi_n \, dq
\]

\[
= \int \phi^* n (\hat{P} \phi^* m) \, dq = \int \phi^* m (\hat{P} \phi n) \, dq \quad \text{by def}
\]

Therefore

\( (f_n - f_m) \int \phi^* n \phi^* m \, dq = 0 \)

If \( f_n - f_m \neq 0 \) \( \Rightarrow \int \phi^* m \phi n \, dq = 0 \)
If two operators $\hat{f}$ and $\hat{g}$ corresponding to the physical quantities $f$ and $g$ have the same eigenfunctions, then they commute: $[\hat{f}, \hat{g}] = 0$.

Proof:

$$\hat{f} \psi_n = f_n \psi_n \quad \hat{g} \psi_n = g_n \psi_n$$

Let $\psi = \sum_n q_n \psi_n$.

Then

$$(\hat{f} \hat{g}) \psi = \hat{f} (\hat{g} \psi) = \hat{f} \left( \sum_n q_n (g_n \psi_n) \right) = \sum_n q_n \hat{f} (g_n \psi_n) = \sum_n q_n g_n (\hat{f} \psi_n) = \sum_n q_n g_n f_n \psi_n$$

Similarly

$$(\hat{g} \hat{f}) \psi = \sum_n q_n g_n f_n \psi_n$$

$$\Rightarrow (\hat{f} \hat{g} - \hat{g} \hat{f}) \psi = 0 \quad \forall \psi \Rightarrow [\hat{f}, \hat{g}] = 0$$

The converse is also true. Later demonstration.

Function of operators:

Let $\hat{f} \psi_n = f_n \psi_n$.

Then, formally,

$$\hat{f} (\psi) = \psi (0) + \psi' (0) \hat{f} + \frac{\psi'' (0)}{2!} \hat{f}^2 + \cdots$$

$$\Rightarrow \Psi (\hat{f}) \psi_n = \left[ \psi (0) \psi_n + \psi' (0) \frac{f_n}{2!} + \cdots \right] \psi_n = \Psi (f_n) \psi_n$$
Then

\( \psi_f \psi = \sum_n a_n \psi_{f_n} \psi_n \)

\[ = \sum_n a_n \psi(f_n) \psi_n(q) \]

\[ a_n = \int \psi_{f_n}^* \psi dq \]

\[ = \int \left[ \sum_n \psi(f_n) \psi_n(q) \psi_n^*(q) \right] \psi(q) dq \]

\[ = \int K_{\psi}(q, q') \psi(q') dq' \]

where

\[ K_{\psi}(q, q') = \sum_n \psi(f_n) \psi_n(q) \psi_n^*(q') \]

Note that this definition does not require an actual power expansion

- Product of operators -

Remember that

\[ \int \psi(fu) dq = \int \psi(f^T u) dq \]

Then I can write.

\[ \int \psi \left( \frac{\hat{\phi}}{\phi} \right) dq = \int \psi \left( \frac{\hat{\phi}}{\phi} \right) \left( \frac{\hat{\phi}}{\phi} \right) dq \]

However

\[ \int \psi \left( \frac{\hat{\phi}}{\phi} \right) \left( \frac{\hat{\phi}}{\phi} \right) dq = \int \psi \left( \frac{\hat{\phi} \hat{T}}{\phi} \right) \left( \frac{\hat{\phi} \hat{T}}{\phi} \right) dq = \int \phi \left( \frac{\hat{\phi} \hat{T}}{\phi} \hat{T} \right) \]

That is:

\[ \int \psi \left( \frac{\hat{\phi}}{\phi} \right) dq = \int \phi \left( \frac{\hat{\phi} \hat{T}}{\phi} \hat{T} \right) \]
If I def: \( \hat{f} \hat{g} = \hat{h} \)

Then: \( \hat{h}^T = (\hat{f} \hat{g})^T = \hat{g}^T \hat{f}^T \)

If I took the conj: \( (\hat{f} \hat{g})^* = \hat{g}^* \hat{f}^* \)

Therefore: \( (\hat{f} \hat{g})^* = \hat{f} \hat{g} \) only if \( [\hat{f}, \hat{g}] = 0 \)

In general: 
\[
\hat{f} \hat{g} = \frac{1}{2} (\hat{f} \hat{g} + \hat{g} \hat{f}) + \frac{1}{2} (\hat{f} \hat{g} - \hat{g} \hat{f})
\]

\[\text{Hermitian} \quad \text{Anti-Hermitian}\]